ON THE DUALITY OF GENERALISED EULER FORMULA FOR EUCLIDEAN HYPERSURFACES

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ABSTRACT

In order to define the generalised Euler formula in a dual manner, we studied the angles between two hyperplanes in $\mathbb{R}^{n+1}$ and we obtained that the Gauss curvature can be expressed by the normal curvature and its dual form.

I. INTRODUCTION

In Euclidean space $\mathbb{R}^{n+1}$ of dimension $n + 1$ we consider an $n$-dimensional hypersurface $M$ given by a local coordinate system $\{u^1, u^2, \ldots, u^n\}$. Let $\{x_1, x_2, \ldots, x_{n+1}\}$ be an orthogonal coordinate system of $\mathbb{R}^{n+1}$. We assume that the $x_i$'s are $C^\infty$ functions of $u^a$'s and that $1 \leq i \leq n + 1$, $1 \leq a \leq n$. Let $X$ be a vector whose orthogonal components are $(x_1, \ldots, x_{n+1})$, then the hypersurface $M$ can be characterized by a vector function

$$X = X(u^a), \quad a = 1, \ldots, n.$$ (I.1)

Let us denote by $N$ the unit normal vector field of the hypersurface $M$, then it satisfies the conditions $\langle N, N \rangle = 1$ and $\langle N, \frac{\partial X}{\partial u^a} \rangle = 0$. Now let us introduce an orthonormal frame in $\mathbb{R}^{n+1}$ by $e_i$, and using this frame we can write that

$$N = \sum_{i=1}^{n+1} N_i e_i$$ (I.2)

and that

$$\frac{\partial X}{\partial u^k} = \sum_{i=1}^{n+1} (x_k)_i e_i, \quad k = 1, \ldots, n,$$ (I.3)

where $N = N_i (u^a)$, $a = 1, \ldots, n$, $1 \leq i \leq n + 1$. 
II. PRELIMINARIES

Let \( v \) denote a tangent vector of the tangent space \( T_M(m) \) at the point \( m \) of hypersurface \( M \). In this direction the curvature \( \frac{1}{R} \) of the hypersurface \( M \) is defined by

\[
\frac{1}{R} = - \langle v, \frac{\partial N}{\partial u^k} \rangle = h_{x\beta} u^x u^\beta \quad \text{(II.1)}
\]

where \( h_{x\beta} \) is the second fundamental tensor of \( M \) and defined as

\[
h_{x\beta} = \langle N, \frac{\partial^2 X}{\partial u^x \partial u^\beta} \rangle = - \langle \frac{\partial N}{\partial u^x}, \frac{\partial X}{\partial u^\beta} \rangle.
\]

The principal curvatures at a point of \( M \) are the eigenvalues of the second fundamental tensor evaluated at this point. Hence they are the roots of the characteristic equation as follows

\[
det \left[ h_{x\beta} - \frac{1}{R} g_{x\beta} \right] = (-1)^n \det (g_{x\beta}) \left( \frac{1}{R} - \frac{1}{R_1} \right) \cdots \left( \frac{1}{R} - \frac{1}{R_n} \right)
= (-1)^n \det (g_{x\beta}) \left\{ \frac{1}{R^n} - \frac{1}{R^{n-1}} \left( \sum_{i_1=1}^{n} \frac{1}{R_{i_1}} \right) + \frac{1}{R^{n-2}} \left( \sum_{i_1 < i_2} \frac{1}{R_{i_1} R_{i_2}} \right) 
+ \cdots + (-1)^{n-1} \frac{1}{R} \left( \sum_{i_1 < \ldots < i_{n-1}} \frac{1}{R_{i_1} \ldots R_{i_{n-1}}} \right) + (-1)^n \frac{1}{R_1 \ldots R_n} \right\} = 0 \quad \text{(II.2)}
\]

where \( g_{x\beta} \)'s are the coefficients of the first fundamental form of the hypersurface \( M \). The principal directions always exist and we can find an orthonormal system of principal directions.

Now let \( \theta_x \) denote the angles between the direction \( v \) and the principal directions, where \( \alpha \) runs from 1 to \( n \). If we denote the principal directions by \( t_1, \ldots, t_n \), then \( \theta_1 \triangleleft (t_1, v), \ldots, \theta_n \triangleleft (t_n, v) \).

The curvature \( \frac{1}{R} \) in this direction \( v \) can be expressed in terms
of the principal curvatures \( \frac{1}{R_i} \), \( i = 1, \ldots, n \), by means of Euler's formula

\[
\frac{1}{R} = \sum_{i=1}^{n} \frac{1}{R_i} \sin^2 \theta_i. \tag{II.3}
\]

Now let us define a kind of normal curvature which we will denote by \( \overline{R} \) and will be thought as a dual corresponding of \( R \). This will be defined at the image point of \( m \) under the normal projection in the direction \( v \) of \( M \). This concept has been defined by A. Mannheim (see [1] and [4]). From this dual viewpoint the Euler formula may be constructed as

\[
\overline{R} = \sum_{i=1}^{n} R^*_i \sin^2 \theta_i \tag{II.4}
\]

where \( R^*_i \) shows the dual principal curvature corresponding to \( \frac{1}{R_i} \).

Denoting by \( v \) the rectangular components of the unit vector \( v \) we write that

\[
v = \sum_{i=1}^{n} v_i e_i. \tag{II.5}
\]

Also we have that \( \cos \theta_i = \langle v, e_i \rangle \), \( i = 1, \ldots, n \). On multiplying both members of (II.5) by \( e_k \) we find that \( v_k = \langle v, e_k \rangle \) and that \( v_i = \cos \theta_i, i = 1, \ldots, n \). Consequently we have

\[
v = \sum_{i=1}^{n} e_i \cos \theta_i \quad \text{or} \quad \sum_{i=1}^{n} \cos^2 \theta_i = 1. \tag{II.2}
\]

III. ANGLES BETWEEN HYPERPLANES IN \( R^{n+1} \)

Let us consider two \( n \)-dimensional tangent vectors \( T_1^n, T_2^n \) which are \( n \)-planes in euclidean space \( R^{n+1} \) and \( t_1 \) and \( t_2 \) be the tangent vectors of the normal sections of \( T_1^n \) and \( T_2^n \) with hypersurface \( M \). Also define the angles between the vectors \( t_1, t_2 \) and any vector in tangent space \( T_m(m) \). To find this angles we will follow the procedure which has been given by H. Gluck [2]. The angle between a pair of lines in euclidean space \( R^{n+1} \) is the smaller of the two possible angles between any vectors parallel to these lines. The angle between a line and a hyperplane (that will be consider as a tangent vector to \( M \)) is
the smallest angle between this line and any line in hyperplane. This is the same as the angle between a line and its orthogonal projection in hyperplane, or $\pi/2$ in case this orthogonal projection degenerates to a point. Let us consider now a pair of hyperplanes of $n$-dimensional $T_1^n$ and $T_2^n$ in $\mathbb{R}^{n+1}$. Suppose that among all pairs of lines, one from $T_1^n$ and one from $T_2^n$, the lines $t_1$ and $t_2$ make the smallest possible angle, $w_1$, with each other. Let $T_1^{n-1}$ and $T_2^{n-1}$ be the orthogonal complements of $t_1$ and $t_2$ in $T_1^n$ and $T_2^n$, respectively. Then it is easily seen that $t_1$ is orthogonal not only to $T_1^{n-1}$ but also to $T_2^{n-1}$, and similarly $t_2$ is orthogonal not only to $T_2^{n-1}$ but also to $T_1^{n-1}$. If we iterate this procedure with $T_1^{n-1}$ and $T_2^{n-1}$ in the roles of $T_1^n$ and $T_2^n$, we get another angle $w_2 = w_1$. Doing this $n$-times we get $n$ angles $0 \leq w_1 \leq w_2 \leq \ldots \leq w_n \leq \pi/2$. This angles depend only on $T_1^n$ and $T_2^n$, and not on the various choices possible during the above procedure, and these angles are called the principal angles between the hyperplane $T_1^n$ and $T_2^n$. If we choose two orthonormal bases $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ for the subspaces $V_1^n$ and $V_2^n$ parallel to $T_1^n$ and $T_2^n$ such that $\langle u_i, v_i \rangle = \cos w_i$ for $1 \leq i \leq n$ and $\langle u_i, v_j \rangle = 0$ for $i \neq j$.

Note that the orthogonal projection of $v_i$ into $V_2^n$ is $(\cos w_i) v_i$ and the orthogonal projection of $v_i$ into $V_1^n$ is $(\cos w_i) u_i$. Suppose that it is desired to find a single angle which might reasonably be called the angle between $T_1^n$ and $T_2^n$. If one is forced to choose from among the principal angles, one would have to select the largest principal angle $w_n$ for such a role, in order to insure that $T_1^n$ and $T_2^n$ are parallel if and only if the angle between them is zero. To arrive at the right definition carefully consider the case $n = 1$.

Then there is just one principal angle $w$ between the lines $t_1$ and $t_2$ and it coincides with the ordinary angle $\theta$ between these lines. This angle $\theta$, lying between $0$ and $\pi/2$, has the following property. If $U$ is any measurable subset of $t_1$ with one-dimensional measure $s(U)$, then the orthogonal projection of $U$ into $t_2$ is also measurable and has one-dimensional measure $(\cos \theta) s(U)$ in $t_2$. Similarly, if $U'$ is a measurable subset of $t_2$ with measure $s(U')$, then the orthogonal projection of $U'$ into $t_1$ has measure $(\cos \theta) s(U')$ in $t_1$. Thus the angle $\theta$ between $t_1$ and $t_2$ may be defined as that angle between $0$ and $\pi/2$ whose cosine is the reduction factor for one-dimensional measure under orthogonal projection of $t_1$ into $t_2$, then (via) matrix of the orthogonal projection of $V_{t_1}$ into $V_{t_2}$ has a determinant whose absolute value is $\cos \theta$. So we can give the following definition directly.
**Definition III.1.** Let $T^1_n$ and $T^2_n$ be hyperplanes in $R^{n+1}$. Let the number $p$, $0 \leq p \leq 1$, be the reduction factor for $n$-dimensional measure under orthogonal projection of $T^1_n$ into $T^2_n$. Then the unique angle $\theta$, $0 \leq \theta \leq \pi/2$ such that $\cos \theta = p$, will be called the angle between $T^1_n$ and $T^2_n$. The following theorem gives us the relation between the angle $\theta$ and the principal angles $w_1, \ldots, w_n$.

**Theorem III.1.** Let $T^1_n$ and $T^2_n$ be hyperplanes in $R^{n+1}$, and let $w_1 \leq w_2 \leq \ldots \leq w_n$ be the principal angles between them. Then the angle $\theta$ between $T^1_n$ and $T^2_n$ is given by $\cos \theta = \cos w_1 \ldots \cos w_n$. For a proof of this theorem see the paper [2]. To give a practical technique for computing the angle between two hyperplanes we will express the following theorem:

**Theorem III.2.** Let $T^1_n$ and $T^2_n$ be hyperplanes in $R^{n+1}$, and let $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ be arbitrary bases for $V^1_n$ and $V^2_n$, respectively. Then the angle $\theta$ between the hyperplanes is given by the formula

$$\cos \theta = \frac{\left| \det (u_i, v_j) \right|}{\sqrt{\det (u_i, u_j)} \sqrt{\det (v_i, v_j)}}. \tag{III.1.}$$

This formula is the generalisation of the formula known for one dimensional two vectors in a vector space. Now we will give a classical concept which is called Dupin indicatrix.

**Definition III.2.** Dupin indicatrix $I_m$ at each point $m$ in $M$ is the subset of $T^0_d(m)$ consisting of all vectors $z$ such that $<S_z, z> = \pm 1$ and $S_z = \hat{D} z : N$, where $S$ is the weingarten map and $\hat{D}$ is the natural connection defined on $R^{n+1}$, [3]. Now let $t_1, \ldots, t_n$ be an orthonormal set of eigen vectors of the map $S^*$ which will be assumed as dual corresponding of the weingarten map $S$. Then $z = \sum_{j=1}^{n} a_j t_j$ and we write

$$<S^*z, z> = \sum_{j=1}^{n} a_i S^* t_i, \sum_{j=1}^{n} a_i t_j = \sum_{j=1}^{n} (a_i)^2 <S^* t_i, t_j>$$

$$= \sum_{j=1}^{n} (a_i)^2 \frac{1}{R^*_1 \ldots R^*_j \ldots R^*_n} \tag{III.2}$$

where $\hat{R}^*_j$ indicates that the $R^*_j$ is omitted as an argument.

Now let $Q$ be a point in the intersection of $I_m$ and $T^0_o$, then we illustrate the following figure in dimension 2.
By using figure III.1, for n-dimension we might infer that

\[ a^j = \frac{\sqrt{R} \sin \theta_j}{\sin \left( \sum_j \theta_j \right)} , \quad 1 \leq j \leq n, \quad (\text{III.3.}) \]

where \( \theta_j \)'s are defined as in the Theorem III.1. Putting (III.3) into (III.2) we get from (II.4) that

\[
\frac{\sin^2 \left( \sum \theta_j \right)}{R} = \frac{\sin^2 \theta_1}{R^*_1 R^*_2 \ldots R^*_n} + \frac{\sin^2 \theta_2}{R^*_1 R^*_2 \ldots R^*_n} + \ldots \\
+ \frac{\sin^2 \theta_n}{R^*_1 \ldots R^*_{n-1} R^*_n} \quad (\text{III.4})
\]

CONCLUSION

For the special case \( \sum \theta_j = \pi / 2 \) we have \( R_i = R_i^* \), \( 1 \leq i \leq n \), so the expression (III.4) changes into (II.3), but there is a slight difference that we will omit it here. (III.4) gives us a dual form of generalised Euler formula. Thus we get a relation between \( R \) and \( \bar{R} \) by using (III.4) and (II.4). To get this we will use that

\[ R^*_1 \ldots \hat{R}^*_j \ldots R^*_n = \frac{1}{K R_j^*} , \quad (K \text{ is the Gaussian curvature}), \]

so we have that
$$R \overline{R} = \frac{\sin^2(\sum \theta_j) \left( R_1^* \sin^2 \theta_1 + \ldots + R_n^* \sin^2 \theta_n \right)}{\frac{1}{K R_1^*} + \ldots + \frac{1}{K R_n^*}}$$

or $K = \frac{\sin^2(\sum \theta_j)}{R \overline{R}}$. And finally for the special case $\sum \theta_j = \pi/2$

we find that

$$K = \frac{1}{R \overline{R}}.$$

(III.5)

REFERENCES


