ON THE FOCAL SURFACES OF THE CONGRUENCES GENERATED BY THE INSTANTANEOUS SCREWING AXES CONNECTED WITH SOME SURFACES

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ABSTRACT

In this paper, the focal surfaces of the congruences derived in [1] and [3] have been investigated and correspondences between them have been explained.

1. INTRODUCTION

Let a surface \( \vec{x} \) be referred to its lines of curvatures. The congruences generated by the instantaneous screwing axes \( \vec{G}, \vec{G}_* \) of the moving trihedrons connected with these lines are respectively

\[
\begin{align*}
\begin{cases}
\vec{y} = \vec{r} + t \vec{g}, & \vec{r} = \vec{x} + \frac{1}{r} \vec{z} \\
\vec{z} = \vec{z} & \\
\vec{y} = \vec{r} + t \vec{g}, & \vec{r} = \vec{x} + \frac{1}{r} \vec{z}
\end{cases}
\end{align*}
\]

(1.1)

[1] In case \( \vec{y} \) and \( \vec{y} \) are normal congruences, let the surfaces generating these, be \( \vec{z} \) and \( \vec{z} \). And let these surfaces be referred to their lines of curvature. The congruence generated by the instantaneous screwing axis \( \vec{G}_* \) of the moving trihedron connected with the lines of curvature \( u = \text{const.} \) of \( \vec{z} \) are

\[
\begin{align*}
\vec{y}^* = r + t^* \vec{g}^*, & \vec{r} = \vec{z} + \frac{1}{b} \vec{n}.
\end{align*}
\]

(1.2)
And the congruence generated by the instantaneous screwing axis \( \mathbf{G}^{**} \) of the moving trihedron connected with the lines of curvature \( v = \text{const. of } z \) are

\[
\begin{align*}
\nabla y^{**} &= \nabla x + t^{**} \nabla g^{**}, \quad \nabla r = \nabla z + \frac{1}{\beta} \nabla n
\end{align*}
\]

(1.3)

[3].

2. THE PROPERTIES OF THE FOCAL SURFACES OF THE CONGRUENCES \( y, y, y^*, y^{**} \)

Since \( p, k \) are the focal surfaces of the congruence \( y; p, k \) of \( y; p^*, k^* \) of \( y^*; \) and \( p^{**}, k^{**} \) of \( y^{**} \) [3], to investigate considering the cases where they coincide and refer to their lines of curvature, first we may write the moving trihedrons (DARBOUX's trihedrons) connected with a common point before calculating their first and second fundamental forms.

1) Since the moving trihedron connected with the point \( x \) of the line of curvature \( v = \text{const. on the surface } \hat{x}(u, v) \) is \( (\hat{x}_1, \hat{x}_2, \hat{z}) \), the trihedrons connected with the focal points corresponding to \( \mathbf{p}_{11} \) of the focal surfaces \( p, p^*, k^{**} \) belonging to the congruences \( y, y^*, y^{**} \) and coinciding with the center surface \( r \) of the surface \( \hat{x} \), are respectively,

\[
(\hat{z}, \hat{x}_2, -\hat{x}_1), \quad (\hat{z}, \hat{x}_2, -\hat{x}_1), \quad (-\hat{z}, -\hat{x}_2, -\hat{x}_1).
\]

2) Since the moving trihedron connected with the point \( \hat{x} \) of the line of curvature \( u = \text{const. on the surface } \hat{x}(u, v) \) is \( (\hat{x}_2, -\hat{x}_1, \hat{z}) \), the trihedrons connected with the focal points corresponding to \( \mathbf{p}_{11} \) of the focal surfaces \( p, k^*, p^{**} \) belonging to the congruences \( y, y^*, y^{**} \) and coinciding with the center surface \( r \) of the surface \( \hat{x} \), are respectively,
\[ (\xi, -x_1, -x_2), \quad (\bar{\xi}, -x_1, -x_2), \quad (-\bar{\xi}, x_1, -x_2). \]

If we calculate the first and the second fundamental forms of the above focal surfaces

1) for the local surfaces \( \vec{p}, \vec{p}^*, \vec{k}^{**} \), we find,

\[
E_{II} = \bar{F}^{**}_{II} = \bar{E}^{**}_{I} = \left( \frac{1}{r} \right)_1^2 E
\]

\[
F_{II} = \bar{F}^{*}_{II} = \bar{F}^{**}_{I} = \left( \frac{1}{r} \right)_1 \left( \frac{1}{r} \right)_2 \sqrt{EG}
\]

\[
G_{II} = \bar{G}^{*}_{II} = \bar{G}^{**}_{I} = \left( \frac{1}{r} \right)_2 \frac{r^2 + q^2}{q^2} G,
\]

\[ [I]_{II} = [I^*]_{II} = [I^{**}]_I = \left( \frac{1}{r} \right)_1^2 E \, du^2 + 2 \left( \frac{1}{r} \right)_1 \left( \frac{1}{r} \right)_2 \sqrt{EG} \, du \, dv \]

\[ + \left( \frac{1}{r} \right)_2^2 \frac{r^2 + q^2}{q^2} G \, dv^2 \]  

and

\[
L_{II} = \bar{L}^{**}_{II} = \bar{L}^{**}_{I} = \left( \frac{1}{r} \right)_1 r \, E
\]

\[
M_{II} = \bar{M}^{*}_{II} = \bar{M}^{**}_{I} = 0
\]

\[
N_{II} = \bar{N}^{**}_{II} = \bar{N}^{**}_{I} = \left( \frac{1}{r} \right)_2 \frac{rq}{q} \, G,
\]

\[ [II]_{II} = [\bar{I}^{**}]_{II} = [\bar{I}^{**}]_I = r \left[ \left( \frac{1}{r} \right)_1 E \, du^2 + \left( \frac{1}{r} \right)_2 \frac{q}{q} G \, dv^2 \right]. \]

From these we may derive the below conclusion:

**Conclusion: 2.1.** The focal surfaces \( \vec{p}, \vec{p}^*, \vec{k}^{**} \) of the congruences \( \vec{y}, \vec{y}^*, \vec{y}^{**} \) are different positions of the center surface \( \vec{r} \) of the base surface \( \vec{x} \), in space.
Also, the Gaussian and the mean curvature of these focal surfaces, we find

\[ K_{\Pi} = \bar{K}^*_{\Pi} = \bar{K}_1^{**} = \frac{q\bar{q}}{\left(\frac{1}{r}\right)_1 \left(\frac{1}{r}\right)_2} \quad (2.5) \]

and

\[ H_{\Pi} = \bar{H}_{\Pi} = \bar{H}_1^{*} = \frac{\left(\frac{1}{r}\right)_1 q\bar{q} - \left(\frac{1}{r}\right)_2 (r^2 + q^2)}{2r \left(\frac{1}{r}\right)_1 \left(\frac{1}{r}\right)_2} . \quad (2.6) \]

Since, \( r_1 \neq 0, r_2 \neq 0 \) from (2.1) and (2.3) we derive the conditions \( F_{\Pi} = \bar{F}^*_{\Pi} = \bar{F}_1^{**} \neq 0 \) and \( M_{\Pi} = \bar{M}^*_{\Pi} = \bar{M}_1^{**} = 0 \). From these, the following theorem may be stated:

**Theorem 2.2.** Since the surface \( \vec{x}(u, v) \) cannot be a canal surface or at the same time cannot be both Mulür surface and tube-shaped canal surface, the parameter curves \( v = \text{const.} \) and \( u = \text{const.} \) of the focal surfaces \( \vec{p}, \vec{p}^*, \vec{k}^{**} \) of the congruences \( \vec{y}, \vec{y}^*, \vec{y}^{**} \) cannot be the lines of curvature.

Since \( q \neq 0, \bar{q} \neq 0, r_1 \neq 0, r_2 \neq 0 \) in (2.5) and (2.6), we find \( K_{\Pi} = \bar{K}^*_{\Pi} = \bar{K}_1^{**} \neq 0 \) and \( H_{\Pi} = \bar{H}^*_{\Pi} = \bar{H}_1^{**} \neq 0 \).

Therefore the theorem below may be stated:

**Theorem 2.3.** Since the surface \( \vec{x}(u, v) \) cannot be Mulür surface, canal surface or tube-shaped surface, the focal surfaces \( \vec{p}, \vec{p}^*, \vec{k}^{**} \) of the congruences \( \vec{y}, \vec{y}^*, \vec{y}^{**} \) respectively, cannot be developable surface, minimal surface.

2) For the local surfaces \( \vec{p}, \vec{k}^*, \vec{p}^{**} \) we find,
\[ \mathcal{E}_{\Pi} = \mathcal{E}^*_{\Pi} = \mathcal{E}^{**}_{\Pi} = \left( \frac{1}{\mathcal{r}} \right)^2 \frac{\mathcal{r}^2 + \mathcal{q}^2}{\mathcal{q}^2} \mathcal{E} \]

\[ \mathcal{F}_{\Pi} = \mathcal{F}^*_{\Pi} = \mathcal{F}^{**}_{\Pi} = \left( \frac{1}{\mathcal{r}} \right)_1 \left( \frac{1}{\mathcal{r}} \right)_2 \sqrt{\mathcal{E} \mathcal{G}} \]

\[ \mathcal{G}_{\Pi} = \mathcal{G}^*_{\Pi} = \mathcal{G}^{**}_{\Pi} = \left( \frac{1}{\mathcal{r}} \right)^2 \mathcal{G} \]

\[ [\mathcal{I}]_{\Pi} = [\mathcal{I}^*]_{\Pi} = [\mathcal{I}^{**}]_{\Pi} = \left( \frac{1}{\mathcal{r}} \right)^2 \frac{\mathcal{r}^2 + \mathcal{q}^2}{\mathcal{q}^2} \mathcal{E} \ du^2 + 2 \left( \frac{1}{\mathcal{r}} \right)_1 \left( \frac{1}{\mathcal{r}} \right)_2 \sqrt{\mathcal{E} \mathcal{G}} \ du \ dv + \left( \frac{1}{\mathcal{r}} \right)^2 \mathcal{G} \ dv^2 \] (2.8)

and

\[ \mathcal{L}_{\Pi} = \mathcal{L}^*_{\Pi} = \mathcal{L}^{**}_{\Pi} = \left( \frac{1}{\mathcal{r}} \right)_1 \frac{\mathcal{r} \mathcal{q}}{\mathcal{q}} \mathcal{E} \]

\[ \mathcal{M}_{\Pi} = \mathcal{M}^*_{\Pi} = \mathcal{M}^{**}_{\Pi} = 0 \]

\[ \mathcal{N}_{\Pi} = \mathcal{N}^*_{\Pi} = \mathcal{N}^{**}_{\Pi} = \left( \frac{1}{\mathcal{r}} \right)_2 \mathcal{r} \mathcal{G} \]

\[ [\mathcal{I}]_{\Pi} = [\mathcal{I}^*]_{\Pi} = [\mathcal{I}^{**}]_{\Pi} = \mathcal{r} \left[ \left( \frac{1}{\mathcal{r}} \right)_1 \frac{\mathcal{q}}{\mathcal{q}} \mathcal{E} \ du^2 + \left( \frac{1}{\mathcal{r}} \right)_2 \mathcal{G} \ dv \right] \]. (2.10)

From these we may write the below conclusion:

**Conclusion 2.4.** The focal surfaces \( \mathcal{p}, \mathcal{k}^*, \mathcal{p}^{**} \) of the congruences \( \mathcal{y}, \mathcal{y}^*, \mathcal{y}^{**} \) respectively, are different positions of the center surface \( \mathcal{r} \) of the base surface \( \mathcal{x} \), in space.

Also, the values of \( K \) and \( H \) for these focal surfaces are found as

\[ K_{\Pi} = K_{\Pi}^* = K_{\Pi}^{**} = \frac{\mathcal{qq}}{\left( \frac{1}{\mathcal{r}} \right)_1 \left( \frac{1}{\mathcal{r}} \right)_2} \] (2.11)
and

\[
\overline{H}_\Pi = \overline{H}_1^* = \overline{H}_\Pi^{**} = \frac{\left(\frac{1}{\bar{r}}\right)^2 qq^{-} \left(\frac{1}{r}\right)^1 \left(r^2 + q^2\right)}{2\bar{r} \left(\frac{1}{\bar{r}}\right)^1 \left(\frac{1}{r}\right)^2} \quad (2.12)
\]

Since \(\bar{r}_1 \neq 0, \bar{r} \neq 0\) from (2.7) and (2.9), we find the conditions \(\overline{F}_\Pi = \overline{F}_1^* = \overline{F}_\Pi^{**} \neq 0\) and \(\overline{M}^*_\Pi = \overline{M}_1^* = \overline{M}_\Pi^{**} = 0\).

From these we may write the below theorem:

**Theorem 2.5.** Since the surface \(x(u, v)\) cannot be canal surface or tube-shaped surface, the parameter curves \(v = \text{const.}\) and \(u = \text{const.}\) of the focal surfaces \(p, k^*, p^{**}\) of the congruences \(y, y^*, y^{**}\) cannot be lines of curvature.

Since \(q \neq 0, \bar{q} \neq 0\), \(\bar{r}_1 \neq 0, \bar{r}_2 \neq 0\) in (2.11) and (2.12), therefore the below theorem may be written.

**Theorem 2.6.** Since the surface \(x(u, v)\) cannot be Mulür surface or general cylindric surface, canal surface or tub-shaped surface, the focal surfaces \(p, k^*, p^{**}\) of the congruences \(y, y^*, y^{**}\) respectively cannot be developable surface, minimal surface.

On the other hand to investigate the focal surface \(k\) and \(\dot{k}\) belonging to the congruences \(\dot{y}\) and \(\ddot{y}\) respectively and coinciding with the center surface of the surfaces \(\dot{z}\) and \(\ddot{z}\) but which do not coincide with the center surfaces \(\dot{r}\) and \(\ddot{r}\) of the surface \(x(u, v)\), first, we may write the moving trihedrons connected with the focal point corresponding to \(\rho_1\) of \(k\), connected with the focal point corresponding to \(\rho_1\) of \(\dot{k}\) before calculating their first and second fundamental forms.
1) For the focal surface \( \vec{k} \), from

\[
\vec{k}_1 = -\left( \frac{1}{b} \right)_1 \vec{g}, \quad \vec{k}_2 = \frac{r_2q}{r_1q-d_1r} \vec{x}_1 - \left( \frac{1}{b} \right)_2 \vec{g}
\]

and

\[
\vec{n}_1 = \frac{\vec{k}_1 \wedge \vec{k}_2}{\sqrt{(\vec{k}_1 \wedge \vec{k}_2)}}, \quad \frac{qx_2-r \vec{z}}{\sqrt{r^2+q^2}} = -\frac{z}{\vec{n}_1}.
\]

the trihedron

\[
(-\vec{g}, \vec{x}_1, \vec{n}_1) \quad (2.13)
\]

is found.

2) For the focal surface \( \vec{k} \), from

\[
\vec{k}_1 = -\left( \frac{1}{\beta} \right)_1 \vec{g} - \frac{r_2q}{q_2r-d_2q} \vec{x}_2, \quad \vec{k}_2 = -\left( \frac{1}{\beta} \right)_2 \vec{g}
\]

and

\[
\vec{n}_1 = \frac{\vec{k}_2 \wedge \vec{k}_1}{\sqrt{\vec{k}_2 \wedge \vec{k}_1}} = \frac{\vec{x}_1 - \vec{z}}{\sqrt{\vec{r}^2+q^2}} = -\frac{z}{\vec{n}_1},
\]

the trihedron is

\[
(-\vec{g}, \vec{x}_2, \vec{n}_1). \quad (2.14)
\]

If we calculate the first and the second fundamental forms of the focal surfaces
1) For \( \mathbf{t} \), we find,

\[
E_1 = \left( \frac{1}{b} \right)_1^2 E \\
F_1 = \left( \frac{1}{b} \right)_1 \left( \frac{1}{b} \right)_2 \sqrt{EG} \\
G_1 = \left[ \left( \frac{r_2q}{r_1q-q_1r} \right)^2 + \left( \frac{1}{b} \right)_2^2 \right] G, \\
\]

\[
[I]_1 = \left( \frac{1}{b} \right)_1^2 E \, du^2 + 2 \left( \frac{1}{b} \right)_1 \left( \frac{1}{b} \right)_2 \sqrt{EG} \, du \, dv + \\
\left[ \left( \frac{r_1q_2}{r_1q-q_1r} \right)^2 + \left( \frac{1}{b} \right)_2^2 \right] G \, dv^2
\]

and

\[
L_1 = - \left( \frac{1}{b} \right)_1 \frac{r_1q_2-q_1r}{r^2 + q^2} E \\
M_1 = 0 \\
N_1 = \left( \frac{1}{b} \right)_1 \frac{r_2q^2}{r_1q-q_1r} G, \\
\]

\[
[II]_1 = - \left( \frac{1}{b} \right)_1 \frac{r_1q_2-q_1r}{r^2 + q^2} E \, du^2 + \left( \frac{1}{b} \right)_1 \frac{r_2q^2}{r_1q-q_1r} G \, dv^2
\]

and also,

\[
K_1 = - \frac{q^2 (r_1q_2-q_1r)^2}{\left( \frac{1}{b} \right)_1 \left( \frac{1}{b} \right)_1 r^2 (r^2 + q^2)^2}, \\
\]

\[
H_1 = \frac{r_1q_2-q_1r}{2 \left( \frac{1}{b} \right)_1 \left( \frac{1}{b} \right)_1^2 r_1q_2 (r^2 + q^2)^2} \left( \frac{1}{b} \right)_1 \left( r^2 + q^2 \right) r_2q^2 \left[ q^2 \left( \frac{1}{b} \right)_1 \right] - \\
r^2 \left( \frac{1}{b} \right)_1^2 - \left( \frac{1}{b} \right)_2^2 (r_1q_2-q_1r) q^2 \right).
\]
2) For \( \vec{k} \),

\[
\begin{align*}
E_I &= \left[ \left( \frac{1}{\beta} \right)_1^2 + \left( \frac{q_2 q}{q_2 \vec{r} - \vec{r}_2 q} \right)_2^2 \right] E \\
\bar{F}_I &= \left( \frac{1}{\beta} \right)_1 \left( \frac{1}{\beta} \right)_2 \sqrt{EG} \\
\bar{G}_I &= \left( \frac{1}{\beta} \right)_2^2 G,
\end{align*}
\]

\[
[I]_I = \left[ \left( \frac{1}{\beta} \right)_1^2 + \left( \frac{q_2 q}{q_2 \vec{r} - \vec{r}_2 q} \right)_2^2 \right] E \ du + 2 \left( \frac{1}{\beta} \right)_1 \left( \frac{1}{\beta} \right)_2 \sqrt{EG} \ dv \\
&\quad + \left( \frac{1}{\beta} \right)_2^2 G \ dv^2
\] (2.22)

and

\[
\begin{align*}
\bar{L}_I &= \left( \frac{1}{\beta} \right)_2 \frac{q_2 q_2^2}{q_2 \vec{r} - \vec{r}_2 q} E \\
\bar{M}_I &= 0 \\
\bar{N}_I &= \left( \frac{1}{\beta} \right)_2 \frac{q_2 q - q_2 \vec{r}}{\vec{r}^2 + q^2} G,
\end{align*}
\]

\[
[II]_I = \left( \frac{1}{\beta} \right)_2 \frac{q_2 q_2^2}{q_2 \vec{r} - \vec{r}_2 q} E \ du + \left( \frac{1}{\beta} \right)_2 \frac{q_2 q q_2 \vec{r}}{\vec{r}^2 + q^2} G \ dv \] (2.24)

and also,

\[
\bar{K}_I = -\frac{q_2 \left( q_2 \vec{r} - \vec{r}_2 \vec{r} \right)^2}{\left( \frac{1}{\beta} \right)_2 \left( \frac{1}{\beta} \right)_2 \vec{r}^2 \left( \vec{r}^2 + q^2 \right)^2},
\] (2.25)

\[
\bar{H}_I = \frac{q_2 \vec{r} - \vec{r}_2 q}{2 \left( \frac{1}{\beta} \right)_2 \left( \frac{1}{\beta} \right)_2 \left( \vec{r}^2 + q^2 \right) \vec{r}^4 \vec{q}^2}
\] (2.26)
\[ \left\{ \text{i} \bar{q}^2 \left( \text{i} \bar{r}^2 + \bar{q}^2 \right) \left( \frac{1}{\beta} \right) \bar{r}^2 - \left( \frac{1}{\beta} \right) \bar{q}^2 \right\} - \left( \frac{1}{\beta} \right)^2 \bar{q}^2 \bar{r} \bar{q} \bar{r} \bar{q}^2 \]

are written.

1) Since the focal surface \( \tilde{k} \) coincides with the center surface belonging to the lines of curvature \( v = \text{const.} \) of the surface \( \tilde{z} \), that is \( \tilde{k} = \tilde{r} + \rho \tilde{g} = \tilde{z} + \frac{1}{b} \tilde{n} = \tilde{z} - \frac{1}{b} \tilde{g} \), we may write,

\[ b = b_1 = -\left[ \left( \tilde{g} \right) \bar{m}_1 \right] - \left( \frac{1}{l} \right) \tilde{m}_1 \left( \tilde{g} \right) = \frac{r_1 q - q_1 r}{r^2 + q^2} \neq 0, \ (r_1 q - q_1 r \neq 0). \quad (2.27) \]

Since \( F_1 \neq 0 \) \((b_1 \neq 0, \ b_2 \neq 0)\) in (2.15) and \( M_1 = 0 \) in (2.17), the below theorem may be written.

Theorem 2.7. Since the surface \( \tilde{z} \) cannot be canal surface or tube-shaped surface, the parameter curves \( v = \text{const.} \) and \( u = \text{const.} \) of the focal surface \( \tilde{k} \) of the congruence \( \tilde{y} \), cannot be lines of curvature.

From (2.19) and (2.20) \( K_1 \neq 0, \ H_1 \neq 0 \) \((q \neq 0, \ r_1 q - q_1 r \neq 0)\) are seen. From this the below theorem may be written.

Theorem 2.8. Since the surface \( \tilde{x} \) \((u, v)\) cannot be Mulür surface and the surface which have the lines of curvature \( v = \text{const.} \), consisting of plane curves, the focal surface \( \tilde{k} \) of the congruence \( \tilde{y} \), cannot be developable surface, minimal surface.

2) Since the focal surface \( \tilde{k} \) coincides with the central surface belonging to the lines of curvature \( u = \text{const.} \) on the surface \( \tilde{z} \) that

is \( \tilde{k} = \tilde{r} + \rho \tilde{g} = \tilde{z} + \frac{1}{\beta} \tilde{n} = \tilde{z} - \frac{1}{\beta} \tilde{g} \), we may write

\[ \beta = \beta_1 = -\left[ \tilde{m}_1 \left( \frac{\tilde{g}^2}{2} \right) \right] = \frac{q_2 \tilde{r} - \tilde{r}_2 q}{\tilde{r}^2 + q^2} \neq 0, \ (q_2 \tilde{r} - \tilde{r}_2 q \neq 0) \quad (2.28) \]
It can be seen that at (2.21), $\overline{F}_1 \neq 0$ ($\overline{\delta}_1 \neq 0, \overline{\beta}_2 \neq 0$). If we take the condition $\overline{M}_1 = 0$ at (2.23) into consideration together with these, we may write the below theorem.

**Theorem 2.9.** Since the surface $\vec{z}$ cannot be canal surface or tube-shaped surface, the parameter curves $v = \text{const.}$ and $u = \text{const.}$ on the focal surface $\vec{k}$ of the congruence $\vec{y}$, cannot be lines of curvature.

From (2.25) and (2.26) $\overline{K}_1 \neq 0, \overline{H}_1 \neq 0$ ($\overline{q} \neq 0, \overline{q}_2 \overline{i} - \overline{i}_2 \overline{q} \neq 0$) are seen. From this below theorem may be written.

**Theorem 2.10.** Since the surface $\vec{x}$ $(u, v)$ cannot be Mulür surface and the surface which have with the lines of curvature $u = \text{const.}$ consisting of plane curves, the focal surface $\vec{k}$ of the congruence $\vec{y}$ cannot be developable surface, minimal surface.

**REFERENCE**


