ON THE MEUSNIER'S THEOREM FOR LORENTZIAN SURFACES

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ABSTRACT

In the present paper we give an analog of the Meusnier's Theorem for Lorentzian surfaces in the Lorentzian space of the dimension 3.

1. INTRODUCTION

By $L^3$ we denote the space $R^3$ endowed with the inner product $<,>$ of index 1 and call it Lorentzian 3--space. In $L^3$ every tangent space of a surface can be considered as a subspace of $L^3$ in a canonical way. Thus if a surface in $L^3$ has the tangent spaces of index 1 then we call the surface Lorentzian as in [4]. In addition, a curve in a Lorentzian surface called time--like, space--like or null whether its velocity vector is, [1].

In the Riemannian case, it is well known that all the curves pass through a point, say $p$, and have common and non asymptotic tangents at the point $p$ have their curvature centers on a unique sphere and also have their curvature circles on another unique sphere. This fact known as the Meusnier's Theorem (see [2]). The essential part of this work devoted to give an analog of this fact in $L^3$.

Let $x: I \rightarrow L^3$ be a unit speed curve in $L^3$ and $X = \dot{x}$, where the notation dot indicates the derivative. If $x$ is a space--like curve then there exist unique orthonormal vectors $X, Y, Z$, and the first and the second curvature functions $k_1, k_2$ from $I$ to $R$ such that

$$<X, X> = 1, <Y, Y> = -1, <Z, Z> = 1,$$

$$<X, Y> = <Y, Z> = <X, Z> = 0,$$

$$D_XX = k_1Y$$

$$D_XY = k_1X + k_2Z$$

$$D_XZ = k_2Y$$

or
\[
\langle X, X \rangle = 1, \quad \langle Y, Y \rangle = 1, \quad \langle Z, Z \rangle = -1,
\]
\[
\langle X, Y \rangle = \langle Y, Z \rangle = \langle X, Z \rangle = 0,
\]
\[
\begin{align*}
D_x X &= k_1 Y \\
D_x Y &= -k_1 X + k_2 Z \\
D_x Z &= k_2 Y
\end{align*}
\]  
(1.2)

where \(Y\) is time–like or space–like. If the curve \(z\) is time–like then the unique orthonormal frame field \(\{X, Y, Z\}\), exists such that

\[
\begin{align*}
\langle X, X \rangle &= -1, \quad \langle Y, Y \rangle = \langle Z, Z \rangle = 1, \\
\langle X, Y \rangle = \langle Y, Z \rangle = \langle Z, X \rangle &= 0,
\end{align*}
\]
\[
\begin{align*}
D_x X &= k_1 Y \\
D_x Y &= k_1 X + k_2 Z \\
D_x Z &= -k_2 Y
\end{align*}
\]  
(1.3)

where \(\{X, Y, Z\}\) called Frenet frame field of \(z\), [3].

We give the notion of curvature center as the following which is just as in the Euclidean case.

**Definition 1.** Let \(z: I \rightarrow L^3\) be a non–null curve and \(\{X, Y, Z\}\), \(k_1\) are the Frenet frame field on \(z\) and the first curvature function of \(z\). The point

\[
C(t) = z(t) + \frac{1}{k_1(t)} Y
\]

is called the curvature center of \(z\) at the point \(z(t)\) and the pseudo 1–sphere centered at the point \(C(t)\) that lay on the plane spanned by \(X\) and \(Y\) called *curvature circle* of \(z\) at the point \(p\).

Now, we recall a definition about plane sections, just as in the case of \(E^3\), [2], as follows:

**Definition 2.** Let \(M\) be a Lorentzian surface in \(L^3\) and \(\Pi\) a plane which passes through a point \(p \in M\). If a tangent vector \(X_p \in T_M(p)\) is in \(\Pi\) then the intersection curve \(M \cap \Pi\) is called the section curve determined by \(X_p\) and if the plane \(\Pi\) is orthogonal to \(T_M(p)\) then the section curve determined by \(X_p\) is called the *normal section curve* determined by \(X_p\).

Finally,
Definition 3. Let $M \in \mathbb{L}^3$ be a Lorentzian surface and $X_P$ is a tangent vector to $M$ at the point $p$. Let us denote a plane through $X_P$ by $\pi$ and the curvature center of the intersection curve of $\pi$ and $M$, that is $M \cap \pi$, by $C_i$. The curve obtained by translating the curvature circle of the intersection curve $M \cap \pi$, at the point $p$, by the vector $\overrightarrow{C_iP}$ called conjugate curvature circle of the intersection curve $M \cap \pi$ at the point $P$.

2. THE MEUSNIER'S THEOREM FOR LORENTZIAN SURFACES

The main theorems are:

Theorem 1. Let $M$ be a Lorentzian surface in $\mathbb{L}^3$ and $p \in M$, $X_p \in T_M(p)$. We assume that $X_p \in T_M(p)$ is not an asymptotic direction on $M$ then

i) The locus of the curvature centers of all the non–null section curves determined by $X_P$ with space–like second Frenet vectors is a pseudosphere

ii) The locus of the fourth vertex point of the parallelogram which constructed with one diagonal $[CC_i]$ and three vertices $P$, $C$, $C_i$ is a pseudo–sphere where $C_i$ and $C$ are the curvature centers of any section curve and the normal section curve determined by $X_p$, respectively.

Theorem 2. Let $M$ be a Lorentzian surface in $\mathbb{L}^3$ and $p \in M$, $X_P \in T_M(p)$. We assume that $X_p \in T_M(p)$ is not an asymptotic direction on $M$. Let the points $C$ and $C_i$ denote the curvature centers of the normal section curve and a section curve determined by $X_p$. Then,

i) All curvature circles of all the non–null section curves determined by $X_P$ with space–like second Frenet vectors lie on a pseudo–sphere centered at the point $C$.

ii) All the conjugate curvature circles of all non–null section curves determined by $X_P$ with time–like second Frenet vectors lie on a pseudo–sphere or a pseudo–hyperbolic space and the center of the pseudo–sphere or the hyperbolic space is the fourth vertex point of the parallelogram which is determined by the vertex points, $p$, $C$ and $C_i$ and one diagonal the line segment $[CC_i]$.

First of all we shall give the following Lemma.

Lemma 1. Let $h$ be the second fundamental form of the Lorentzian surface $M$ in $\mathbb{L}^3$. If $X_p$ is a tangent vector to $M$ and $V$ and $k_1$ are
the second Frenet vector and the first curvature function of the section curve determined by \( X_p \), respectively. Then
\[
    k_2(0) \ < V_p, N_p > = - h \ (X_p, X_p) \tag{2.1}
\]
where \( N_p \) is the unit normal to \( M \) at the point \( p \).

**Proof** is the same as in the \( E^3 \), so we don't give it here, (see, [5]).

If we consider the curve mentioned in the Lemma 1. as the normal section curve determined by \( X_p \) then the equation (2.1) becomes
\[
    k_N(0) \ < V_p^N, N_p > = - h \ (X_p, X_p)
\]
where we denote the curvature of that normal section curve \( \alpha_N \) by \( k_N(0) \) thus we get
\[
    \begin{cases}
        h \ (X_p, X_p); \ V_p^N = - N_p; \text{ (that is, } \alpha_N \text{ is bending away from } N_p) \\
        - h \ (X_p, X_p); \ V_p^N = N_p; \text{ (that is, } \alpha \text{ is bending forward } N_p)
    \end{cases} \tag{2.2}
\]
where \( V_p^N \) denotes the second Frenet vector of \( \alpha \).

Now we use the term curvature radius which is the reciprocal of the curvature. So we conclude the following corollary.

**Corollary:** Let \( \alpha: I \longrightarrow M \) be a curve on the Lorentzian manifold \( M \) and \( X_p \) is a non-asymptotic tangent vector to \( M \). If \( g, g \) are the curvature radii of the normal section curve and a section curve determined by \( X_p \), respectively, then
\[
    < V_2, N > = \frac{g}{g_N} = \frac{k_N}{k_1} \text{ when } < V_2^N, N > > 0
\]
\[
    < V_2, N > = \frac{-g}{g_N} = \frac{-k_N}{k_1} \text{ when } < V_2^N, N > > 0
\]
where \( V \) is the second Frenet vector of \( \alpha \) and \( N \) is the unit normal vector field to \( M \) and \( k_1, k_N \) denote the curvatures of \( \alpha \) and the normal section curve determined by \( X_p \).

Finally we need the following two Lemmas for the proof of the Theorem 1 and the Theorem 2.

**Lemma 2.** Let \( A, B \in L^3 \) and the vector \( \overrightarrow{AB} \) is space-like. Then the points \( p \) on the condition that
are lies on a sphere $S^2_7(r)$, where the radius $r$ is a constant and depends on the points $A$ and $B$.

**Proof:** We choose an orthonormal basis $\{e_0, e_1, e_2\}$ for $L^3$ such that $e_0$ is a unit time-like vector. Thus, for any point $p \in L^3$ we have the following coordinate expression

$$\vec{OP} = x_0e_0 + x_1e_1 + x_2e_2$$

and we can identify the point $p$ and the vector $\vec{OP}$ as well as

$$x_0e_0 + x_1e_1 + x_2e_2$$

and $(x_0, x_1, x_2)$. Now, take

$$A = (a_0, a_1, a_2)$$
$$B = (b_0, b_1, b_2)$$
$$P = (x_0, x_1, x_2)$$

so

$$\langle \vec{AB}, \vec{AB} \rangle = -(b_0 - a_0)^2 + (b_1 - a_1)^2 + (b_2 - a_2)^2 > 0.$$  \text{(2.3)}

If the point $p$ satisfies the condition of the Lemma then; a direct computation shows that;

$$(x_0 - (1/2)(a_0 + b_0))^2 + (x_1 - (1/2)(a_1 + b_1))^2 + (x_2 - (1/2)(a_2 + b_2))^2 = c$$

where

$$c = (1/4) (- (b_0 - a_0)^2) + (b_1 - a_1)^2 + (b_2 - a_2)^2) + (1/2) (a_0 + b_0)^2$$

and because of (2.3) the constant $c$ is positive. Thus what we get is that the point $p$ lies on a sphere $S^2_1(\sqrt{c})$.

**Lemma 3:** Let $M$ be a Lorentzian surface in $L^3$. If $p \in M$, $X_p \in T_M(p)$ and $z$ is a section curve determined by $X_p$ such that the second Frenet vector $V_2$ of $z$ is time-like then the vector $\vec{PQ}$ is orthogonal to the vector $\vec{PC}$, where $C$ is the curvature center of $z$ at the point $p$ and $Q$ is the fourth vertex point of the parallelogram determined by the vertices $p$, $C_i$ and $C$ such that $[PQ]$ and $[CC_i]$ are diagonal $s$ of the parallelogram and the point $C$ is the curvature center of the normal section curve determined by $X_p$ at the point $p$. Furthermore $PQ$ is a space like vector (Figure. 1).
Proof:

Let $k_1$ and $k_N$ denote the first curvature of the section curve $\alpha$ and the normal section curve determined by $X_p$, respectively. So, in the case of $\langle V_2^N, N \rangle > 0$, we have the following

$$C_1 = p + \frac{1}{k_1} V_2$$

$$C = p + \frac{1}{k_N} N_p$$

where $N_p$ is the unit normal to $M$ at the point $p$ (Figure. 1) (It should be noticed that if $\langle V_2^N, N \rangle < 0$ then we have to take $N_p = -V_2^N$ that is,

$$C = p - \frac{1}{k_N} N_p$$

thus
\[ \vec{PQ} = \frac{1}{k_1} V_2 + \frac{1}{k_N} N_p \]

and

\[ <\vec{PQ}, \vec{PC}_i> = \frac{1}{k_1^2} <V_2, V_2> + \frac{1}{k_1} \frac{1}{k_N} <N_p, V_2> \]

since \( V_2 \) is a time–like curve and

\[ <N_p, V_2> = \frac{k_N}{k_1} \]

by the corollary of Lemma 1 so what we get is that

\[ <\vec{PQ}, \vec{PC}_i> = 0 \]

or

\[ \vec{PQ} \perp \vec{PC}_i. \]

For the second assertion of the Lemma, since \( \vec{PC}_i \) is a time–like vector and we proved that \( \vec{PV} \perp \vec{PC}_i \) as above, so \( \vec{PQ} \) is a space–like vector that completes the proof.

**Proof of the Theorem 1.** We will take the figure 2 into account and assume that \( <V_2^N, \ N_p> > 0 \), thus

\[ \vec{PC} = \frac{1}{k_N} N_p. \]

In the case of \( <V_2^N, \ N_p> <0 \), we have to take the vector \( \vec{PC} \) as \( -(1/k_N) N_p \). We would not deal with this possibility because, it makes no difference between the proofs that involving the signature of the number \( <V_2^N, \ N_p> \). So we proceed the proof as follows

1) If \( V_2 \) is space–like then by the corollary we obtain

\[ <gV_2 - g_N N_p, \ gV_2> = g^2 - g g_N (g / g_N) = 0. \]

On the other hand

\[ \vec{PC}_i = gV_2 \]

\[ \vec{CC}_i = gV_2 - g_N N_p \]

so

\[ <\vec{PC}_i, \vec{CC}_i> = 0 \]
that completes the proof of the assertion i) because of the Lemma 2 (see. Fig. 1).

ii ) If the second Frenet vector $V_2$ is time–like then;

$$\overrightarrow{PQ} = \overrightarrow{PC} + \overrightarrow{PC_1} = gV_2 + g_{\text{N}}N_p$$

$$\overrightarrow{CQ} = \overrightarrow{CP} + \overrightarrow{PQ} = gV_2$$

and by the corollary we obtain

$$<gV_2 + g_{\text{N}}N_p, gV_2> = -g(g_{\text{N}}/g_{\text{N}})$$

so

$$<\overrightarrow{PQ}, \overrightarrow{OC}> = 0$$

which completes the proof for the assertion ii) because of the Lemma 2.

**Proof of the Theorem 2:** Since $C_i$ and $C$ are curvature centers, we can write
\[ C_i = p + \frac{1}{k_1} V_2 \]

and

\[ C = p + \frac{1}{k_N} N_p \]

where, \( k_1 \) and \( k_N \) are first curvature function of the section and the normal section curve determined by \( X_p \). \( V_2 \) denotes the second Frenet vector of the section curve and \( N_p \) is the unit normal to \( M \) at the point \( p \).

On the other hand, \( X_p \) is orthogonal to both \( \overrightarrow{PC} \) and \( \overrightarrow{PC_i} \) so the vector \( \overrightarrow{CC_i} \) orthogonal to the vectors \( X_p \) and \( \overrightarrow{PC_i} \) (figure. 3). Thus \( \overrightarrow{CC_i} \) orthogonal to the plane spanned by the vectors \( \overrightarrow{PC_i} \) and \( X_p \) at the point \( p \).

![Figure. 3](image)

(i) Let \( Z \) be a point that lies on the curvature circle at the point \( p \) of the section curve determined by \( X_p \). Since \( \overrightarrow{CC_i} \) is orthogonal to the plane spanned by \( \overrightarrow{PC_i} \) and \( X_p \) and
\[
\vec{ZC}_1 \in S_p \{X_p, \vec{PC}_1\}
\]

thus

\[
< \vec{ZC}, \vec{ZC} > = < \vec{PC}_1, \vec{PC}_1 > + < \vec{C}_1, \vec{C}_1 > . \tag{2.4}
\]

On the other and:

\[
\vec{PC} = \vec{PC}_1 + \vec{C}_1 \vec{C}
\]

and so

\[
< \vec{PC}, \vec{PC} > = < \vec{PC}_1, \vec{PC}_1 > + < \vec{C}_1 \vec{C}, \vec{C}_1 \vec{C} > + 2 < \vec{PC}_1, \vec{C}_1 \vec{C} >
\]

since \( \vec{C}_1 \vec{C} \perp \vec{PC}_1 \) thus the right hand side of the above equation is the same as the right hand side of the equation (2.4) so

\[
< \vec{PC}, \vec{PC} > = < \vec{ZC}, \vec{ZC} >
\]

which means that, the point Z lies on the pseudo–sphere centered at the point C. Since Z is arbitrary that completes the proof of the assertion (i).

(ii) We will take the figure. 4 into account so we proceed the proof as follows

\[\text{Figure. 4}\]
Let $Z$ be a point that lies on the special translated curvature circle of the section curve at the point $p$ determined by $X_p$.

By Lemma 3; $\vec{PQ}$ is orthogonal to $\vec{PC_i}$. Since $\vec{PQ}$ is a vector in the plane spanned by $N_p$ and $V_2$ then $\vec{PQ}$ is orthogonal to the vectors $V_2$ and $X_p$ so we obtain

$$\langle \vec{PQ}, \vec{PZ} \rangle = 0$$

so we get

$$\langle \vec{QZ}, \vec{QZ} \rangle = \langle \vec{QP}, \vec{QP} \rangle + \langle \vec{PZ}, \vec{PZ} \rangle.$$  \hspace{1cm} (2.6)

By the Definition 3, there exists a point $Y$ on the curvature circle at the point $p$ determined by $X_p$, such that

$$\vec{YZ} = \vec{C_iP}$$

thus

$$\vec{C_iY} = \vec{PZ}.$$  \hspace{1cm} (2.7)

Taking (2.7) into (2.6) we get

$$\langle \vec{QZ}, \vec{QZ} \rangle = \langle \vec{QP}, \vec{QP} \rangle + \langle \vec{C_iY}, \vec{C_iY} \rangle$$

and since $Y$ is a point on the curvature circle centered at $C_i$ then

$$\langle \vec{C_iY}, \vec{C_iY} \rangle = \langle \vec{PC_i}, \vec{PC_i} \rangle$$

so by (2.8) we obtain

$$\langle \vec{QZ}, \vec{QZ} \rangle = \langle \vec{QP}, \vec{QP} \rangle + \langle \vec{PC_i}, \vec{PC_i} \rangle$$

we recall that $\vec{QP}$ is a space-like, $\vec{PC_i}$ is a time-like so (2.9) can be written as the following form

$$\langle \vec{QZ}, \vec{QZ} \rangle = \| \vec{QP} \|^2 - \| \vec{PC_i} \|^2$$
which completes the proof of the assertion (ii) since the point \( Z \) are lies on a pseudo–sphere or on a pseudo–hyperbolic space according to the sign of the number

\[
\| \overrightarrow{QP} \|^2 - \| \overrightarrow{PC_i} \|^2.
\]

REFERENCES


