THE CHARACTERIZATION OF SCHWARZ THEOREM AND UNIT DISCS

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ABSTRACT

Let \( D = \{ z \in \mathbb{C} : |z| \leq r \} \) be a set and \( A(D) \) be an algebra of bounded analytic functions on \( D \). In this paper taking complex algebra \( R \), we gave the characterization of Schwarz theorem. In the special case \( r = 1 \), we obtained the characterization of Schwarz lemma. Taking \( a \in R \) that satisfies some conditions we gave algebraic characterization of conformal mapping from \( D \) to \( \mathbb{U} \), where \( \mathbb{U} = \{ w \in \mathbb{C} : |w| \leq 1 \} \), and investigate the case \( r = 1 \).

INTRODUCTION

This paper presents a solution to problem in subject of rings of analytic functions. In late 1940's, it was shown that two domains; \( D_1 \) and \( D_2 \) in the complex plane, are conformally equivalent iff the rings \( B(D_1) \) and \( B(D_2) \) of all bounded analytic functions defined on them are algebraically isomorphic. Let \( R \) be a ring. It is well known that if \( R \) is isomorphic with the ring of bounded analytic functions on an annulus \( A = \{ z \in \mathbb{C} : \rho_1 < |z| < \rho_2 \} \), where \( \rho_1 \) and \( \rho_2 \) are unknown, then it deduces the number \( \rho_1 / \rho_2 \) from the ring \( R \) [2].

In our study we have taken the known ring and given some algebraic characterizations.

ALGEBRAIC CHARACTERIZATIONS

Let \( \phi \) be an isomorphism mapping \( B(D) \) onto \( R \). We will denote elements of \( B(D) \) by \( f, g, f, \ldots \) and elements of \( R \) by \( a, b, c, \ldots \). Let \( e \) and \( 1 \) be multiplicative identity of \( R \) and \( B(D) \), respectively. Thus, \( 1 \in B(D) \) is the function identically equal to \( 1 \) on \( D \). Since \( \phi : B(D) \rightarrow R \) is an isomorphism, \( \phi(1) = e \). Furthermore \( \phi(n1) = ne \), so that \( \phi(\pm (m/n) \cdot 1) = \pm (m/n) e \). \(-e \) has two square roots in \( R \), one is
the image of \( i.1 \), the other is the image of \(-i.1\). It is algebraically impossible to distinguish between these, since \( R \) has an automorphism which takes one into the other (corresponding to the mapping \( f \rightarrow \bar{f} \in B(\mathcal{D}) \)). Thus, we choose one root of \(-e\) and make it which correspond to \( i.1\); denote it as \( ie \).

Henceforth, we will denote the complex number field by \( C \) and the complex rational number field by \( C_r \). Where a complex number, both real and imaginary parts are real rationals, is called a complex rational number. Clearly, \( C_r \) and \( C \) are subrings of \( B(\mathcal{D}) \).

**Lemma 2.1.** For each \( z \in C, \mathcal{O}(z) = z \) (or \( \bar{z} \)).

**Proof:** If \( z \in C_r \), there are the rational numbers \( r_1 \) and \( r_2 \) such that \( z = r_1 + ir_2 \). Since \( \mathcal{O}(1) = e \) and \( \mathcal{O}(i) = i(0r - i) \), we get \( \mathcal{O}[(r_1 + ir_2).1] = r_1e + r_2ie \) (or \( r_1e - r_2ie \), \([3], [4]\)).

**Lemma 2.2.** For each real number \( c, \mathcal{O}(c1) = ce \).

**Proof:** If \( c \) is a rational number, by the Lemma (2.1), \( \mathcal{O}(c1) = ce \). If \( c \) is an irrational number, for each rational number \( c, c - r \neq 0 \).

Thus there exist \((c - r)^{-1} = \frac{1}{c - r} \). Then \( \mathcal{O}[(c - r).1] = \mathcal{O}(c1) - re \) and \( \mathcal{O}\left[\left(\frac{1}{c-r}\right)1\right] = \frac{e}{\mathcal{O}(c1)-re} \). Therefore \( \mathcal{O}(c1) = ce \).

**Corollary 2.3.** If \( c \in C, \mathcal{O}(c1) = ce, [2] \).

**Lemma 2.4.** Let \( f \in B(\mathcal{D}) \) and let \( \mathcal{R}_f \) be the closed range of \( f \). Then \( \lambda \in \mathcal{R}_f \) iff \( f - \lambda 1 \) has no inverse in \( B(\mathcal{D}) \).

**Proof:** If \( \lambda \in \mathcal{R}_f \) there is \( z_0 \in \mathcal{D} \) such that \( f(z_0) = \lambda \). Then \((f-\lambda 1)(z_0) = 0 \). Hence \( f - \lambda 1 \) has no inverse in \( B(\mathcal{D}) \). Now we suppose that \( f - \lambda 1 \) has no inverse in \( B(\mathcal{D}) \). Then at least for one point \( z_0 \in \mathcal{D}, (f - \lambda 1)(z_0) = 0 \). If follows that \( f(z_0) = \lambda \), i.e. \( \lambda \in \mathcal{R}_f \).

**Lemma 2.5.** \( \lambda \in \mathcal{R}_f \) iff \( \mathcal{O}(f) - \lambda e \) has no inverse in \( R \).

**Proof:** If \( \lambda \not\in \mathcal{R}_f \), \( f - \lambda 1 \) has no inverse in \( B(\mathcal{D}) \) by Lemma 2.4. Since \( \mathcal{O} \) is an isomorphism, \( \mathcal{O}(f - \lambda 1) = \mathcal{O}(f) - \lambda e \) has no inverse in \( \mathcal{R} \). [1]

Let \( \sigma(f) \) and \( \sigma(a) \) be spectrum of \( f \in B(\mathcal{D}) \) and \( a \in R \) respectively. If

\[ \rho(a) = \sup \{ |\lambda| : \lambda \in \sigma(a) \}, \]
then \( \rho(a) \) is also the maximum modulus (Hereinafter abbreviated MM) of \( \varphi^{-1}(a) \).

In this paper, we always consider complex algebra. Now we give our theorem connected with algebraic characterization.

**Theorem 2.6.** Let \( R \) be a complex algebra, \( a, b, c \in R \) and \( \varphi : B(\bar{D}) \to R \) be a \( C \)-isomorphism. If \( \varphi^{-1}(b) = z \), then \( \rho(a) = M \) algebraically characterizes Schwarz Theorem.

**Proof:** Let \( \varphi^{-1}(c) = \varphi(z) \), where \( b, c \in R \) and \( a = b.c. \) Then \( \varphi^{-1}(a) = f(z) \). Since \( \varphi^{-1}(a) = \varphi^{-1}(b) \varphi^{-1}(c) \), we obtain \( f(z) = z \varphi(z) \). We can write from here

\[
\varphi(z) = \frac{f(z)}{z},
\]

for \( z \neq 0 \).

For \( \varphi(z) \) to be in \( B(\bar{D}) \), \( f(z) \) must be zero at \( z = 0 \), i.e. \( f(0) = 0 \). Because, as \( f(0) = 0 \) the point \( z = 0 \) is a removable singular point for the function \( \varphi(z) \). Hence, for each \( z, \varphi(z) \in B(\bar{D}) \). By the maximum modulus principle in a disk that concentric with \( \bar{D} \) and has a radii \( k < r \),

\[
|\varphi(z)| \leq \frac{M}{k},
\]

because \( \rho(a) = \text{MM} (\varphi^{-1}(a)) = M \). It follows from that for \( k \to r \)

\[
|\varphi(z)| \leq \frac{M}{r}
\]

that is,

\[
|f(z)| \leq \frac{M}{r} |z|.
\]

If we take \( M = 1 \) and \( r = 1 \) as a result of Theorem 2.6, we obtain an algebraic characterization of Schwarz Lemma. More clearly,

**Corollary 2.7.** Let \( R \) be complex algebra \( a, b, c \in R \) and \( \varphi : B(\bar{U}) \to R \) be \( C \)-isomorphism. If \( \varphi^{-1}(b) = z \), then \( \rho(a) = 1 \) algebraically characterizes Schwarz Lemma.

Another result of Theorem 2.6 is the following.

**Corollary 2.8.** Let \( B(D) \) be a complex algebra of the bounded analytic functions on \( D \) and \( f \in B(D) \) be schlicht. Furthermore, suppose that \( f(0) = 0 \) and \( \text{MM}(f) = 1 \). Then,
\[ f(z) = \frac{1}{r} \exp i\theta \cdot z \]

where \( \mathring{D} = \{ z \in \mathbb{C} : |z| \leq r \} \).

**Proof:** Since \( w = f(z) \) schlicht, \( z = f^{-1}(w) \in B(\mathring{U}) \). Then, we deduce

\[ |f(z)| \leq \frac{M}{r} |z| \]

by the Schwarz Theorem.

Since \( f \) is the function from \( \mathring{D} \) to \( \mathring{U} \), we obtain

\[ |f(z)| \leq \frac{1}{r} |z| \]

for \( M = 1 \) and hence \( r |w| \leq |z| \).

Conversely, since the mapping \( z = f^{-1}(w) \) maps the closed unity ball to \( \mathring{D} \), \( M = r \) and \( r = 1 \). Thus,

\[ |f^{-1}(w)| \leq \frac{r}{1} \cdot |w| \]

and from here we get \( |z| \leq r |w| \). We find \( r |w| = |z| \) from both inequalities or

\[ |\frac{w}{z}| = \frac{1}{r} \]

If follows for that

\[ f(z) = \frac{1}{r} \exp i\theta \cdot z \]

The mapping \( f(z) = \frac{1}{r} \exp i\theta \cdot z \) maps \( \mathring{D} \) to \( \mathring{U} \) such that \( f(0) = 0 \).

Now we will give an algebraic characterization of \( f \) which maps conformally \( \mathring{D} \) onto \( \mathring{U} \) such that \( f(z) = 0 \), where \( z \) is interior point of \( \mathring{D} \).

We need the following Lemma.

**Lemma 2.9.** Let \( z \in \mathring{D} \) be. Suppose that \( f \in B(\mathring{D}) \) satisfies the following conditions.
a) \( f(z) = 0 \),

b) \( \text{MM}(f) = 1 \),

c) \( f \) is schlicht.

Then,

\[
f(z) = \lambda \frac{z - \alpha}{r^2 - \bar{z}z}, \tag{2.9.1}
\]

where \( |\lambda| = r \) and \( \bar{D} = \{ z \in \mathbb{C} : |z| \leq r \} \).

**Proof:** \( I_\alpha = \{ f \in B(\bar{D}) : f(\alpha) = 0 \} \) is the maximal ideal of \( B(\bar{D}) \). \( I_\alpha \) is generated by \( h(z) = z - \alpha \), i.e., \( I_\alpha = \langle z - \alpha \rangle \). The function that we are looking for must be in \( I_\alpha \). If \( \alpha = 0 \), by Corollary 2.8 \( f(z) = \lambda \frac{z}{r^2} \). If \( \alpha \neq 0 \), for any \( z \) in \( \bar{D} \) \( \text{MM}(z - \alpha) \neq 1 \). Therefore \( f(z) \neq z - \alpha \). If \( f(z) = (z - \alpha) g(z) \), \( f(\alpha) = 0 \) and \( \text{MM}(f) = 1 \), then \( g(z) \)

must be \( \frac{\lambda}{r^2 - \bar{z}z} \), where \( r = |\lambda| \). Thus

\[
f(z) = \lambda \frac{z - \alpha}{r^2 - \bar{z}z},
\]

where \( r = |\lambda| \).

Furthermore if \( f \) is schlicht, \( f(\alpha) = 0 \) and \( \text{MM}(f) = 1 \), then this function must be in the form of (2.9.1), [5].

**Theorem 2.10.** Let \( R \) be any algebra such that \( \varnothing \) is an isomorphism from \( B(\bar{D}) \) to \( R \). Furthermore, suppose that the following conditions are satisfied for some \( a \in R \).

a) For each \( \lambda \in \sigma(a) = \bar{U} \), there is only one point \( z_0 \).

b) For each \( z \in C, < b - \bar{z}e > \) is a maximal ideal of \( R \). Furthermore, \( \varnothing^{-1}(b) = z \) and \( a \in < b - \bar{z}e > \), where \( b \in R \).

c) \( \varnothing(a) = \text{MM}(\varnothing^{-1}(a)) = 1 \).

Then \( \varnothing^{-1}(a) \) is a conformally mapping from \( \bar{D} \) to \( \bar{U} \) and

\[
\varnothing^{-1}(a) = \lambda \frac{z - \alpha}{r^2 - \bar{z}z},
\]

where \( |\lambda| = r \).
Proof: Since \( a \in < b - xe > \), there is an element \( c \in R \) such that \( (b - xe) c = a \). Since \( \sigma \) is isomorphism, we can write \( \sigma^{-1}(a) = \sigma^{-1}(b - xe) \). \( \sigma^{-1}(c) \) and \( \sigma^{-1}(a) = \{ \sigma^{-1}(b) - \sigma^{-1}(xe) \} \sigma^{-1}(c) \). Thus we find

\[
\sigma^{-1}(a) = (z - \alpha) \sigma^{-1}(c).
\]

By the Lemma 2.9, MM \( (\sigma^{-1}(a)) = 1 \) and hence

\[
\sigma^{-1}(c) = \frac{\lambda}{r^2 - \bar{e}z}.
\]

Clearly, \( \sigma^{-1}(c) \in B(D) \). We obtain

\[
c = \frac{\sigma(\lambda)}{\sigma(r^2) - \sigma(\bar{e}z)} = \frac{\lambda e}{r^2 e - \bar{e}bc}
\]

from the equality and so \( c \in R \). Thus

\[
a = (b - xe) \sigma \left( \frac{\lambda e}{r^2 e - \bar{e}bc} \right) \in (b - xe)
\]

and we deduce the mapping

\[
\sigma^{-1}(a) = \lambda \frac{z - \alpha}{r^2 - \bar{e}z}.
\]

It is well known that this is the mapping from \( D \) onto \( U \). At the same time, the mapping \( \sigma^{-1}(a) \) is unique. Because, \( \lambda_0 \in R \sigma^{-1}(a) \), by \( \lambda_0 \in \sigma(a) \). Since each a point \( R \sigma^{-1}(a) \) correspond to unique \( z_i \) by the Lemma 2.4 and (a), \( \sigma^{-1}(a) \in B(D) \) is one-to-one. Since \( \sigma \) is an isomorphism and \( < b - xe > \) is maximal principal ideal in \( R \), \( \sigma^{-1}(b - xe) \) is a maximal principal ideal in \( B(D) \). This maximal principal ideal is generated by the \( \sigma^{-1}(b) - \sigma^{-1}(xe) = z - \alpha \). Then \( \sigma^{-1}(a) \in < z - \alpha > \) by (b). \( \sigma^{-1}(a) \) is schlicht. Thus

\[
\sigma^{-1}(a) = \lambda \frac{z - \alpha}{r^2 - \bar{e}z},
\]

by Lemma 2.9.

Corollary 2.11. Let \( R \) be any algebra and \( \sigma : B(U) \to R \) be a \( C \)-isomorphism. Furthermore suppose that the following conditions hold.

a) For each \( \lambda_0 \in \sigma(a) = U \), there is an unique \( z_0 \in U \).

b) For each \( \alpha \in C \), \( < b - xe > \) is maximal ideal of \( R \), where \( b \in R \), \( \sigma^{-1}(b) = z \) and \( a \in < b - xe > \).
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c) $\varphi(a) = \text{MM} \left( \varphi^{-1}(a) \right) = 1$.

Then $\varphi^{-1}(a)$ is conformally mapping from $\bar{U}$ onto $U$ and

$$\varphi^{-1}(a) = \lambda \frac{z - \alpha}{1 - \bar{\alpha}z}, \quad (|\lambda| = 1).$$

**Proof:** This corollary is the special case of Theorem 2.10 for $r = 1$.

REFERENCES


