ON A CLASS OF MEROMORPHIC STARLIKE FUNCTIONS WITH POSITIVE COEFFICIENTS

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Let $T_m(A, B, z_0)$ denote the class of functions $f(z) = \frac{a}{z} + \sum_{n=1}^{\infty} a_n z^n$ ($a \geq 1$, $a_n \geq 0$) regular and univalent in the disc $U' = \{z: 0 < |z| < 1\}$, satisfying

$$-z \frac{f'(z)}{f(z)} = \frac{1 + Aw(z)}{1 + Bw(z)},$$

for $z \in U'$, and $w \in E$ (where $E$ is the class of analytic functions $w$ with $w(0) = 0$ and $|w(z)| \leq 1$), where $-1 \leq A < B \leq 1$, $0 \leq B \leq 1$ and $f'(z_0) = -\frac{1}{z_0^2}$ $(0 < z_0 < 1)$. In this paper, sharp coefficient estimates for the class $T_m(A, B, z_0)$ have been studied. Radius of meromorphic convexity, integral transform of functions in $T_m(A, B, z_0)$ have been obtained. It is also proved that the class $T_m(A, B, z_0)$ is closed under a convex linear combination. In the last part, the convolution problem of these functions have been studied.

1. INTRODUCTION

Let $\Sigma$ denote the class of functions of the form

$$h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (1.1)$$

which are regular in $U' = \{z: 0 < |z| < 1\}$ having a simple pole at the origin. Let $\Sigma_S$ denote the class of functions in $\Sigma$ which are univalent in $U$ and $\Sigma^*(\rho)$ be the subclass of functions $f(z)$ in $\Sigma$ satisfying the condition
\[ \text{Re} \left\{ -z \frac{f'(z)}{f(z)} \right\} < \rho. \] (1.2)

Functions in \( \Sigma^*(\rho) \) are called meromorphically starlike functions of order \( \rho \).

The class \( \Sigma^*(\rho) \) have been extensively studied by Pommerenke [5], Clunie [1], Kaczmarski [3], Royster [6] and others.

Let \( \Sigma_M \) denote the subclass of functions in \( \Sigma_s \) of the form \( f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \) with \( a_n \geq 0 \) and let \( \Sigma^*_M(\rho) = \Sigma_M \cap \Sigma^*(\rho) \).

Juneja and Reddy [2] have obtained certain interesting results for functions in \( \Sigma^*_M(\rho) \). Since much work has not been done for meromorphic univalent functions, we introduce following class of functions:

Let \( T_M \) denote the class of functions \( f(z) = \frac{a}{z} + \sum_{n=1}^{\infty} a_n z^n \) (\( a \geq 1, a_n \geq 0 \)) (The \( a \geq 1 \) is necessary, see Nehari [4, Ex. 8, p. 238]) regular and univalent in the disc \( U' = \{ z: 0 < |z| < 1 \} \). Let \( T_M(A, B) \) denote the subclass of functions in \( T_M \) satisfying the condition

\[ -z \frac{f'(z)}{f(z)} \geq \frac{1 - Az}{1 + Bz}, \quad z \in U' \]

where \( z \) denote subordination and A and B are fixed numbers \(-1 \leq A < B \leq 1, 0 \leq B \leq 1 \). Then by definition of subordination

\[ -z \frac{f'(z)}{f(z)} = \frac{1 + A \, w(z)}{1 + B \, w(z)}, \quad \text{for some } z \in U', \ w \in E \] (1.3)

where \( E \) is the class of analytic functions \( w \) with \( w(0) = 0 \) and \( |w(z)| \leq 1 \). Also \( T_M(A, B, z_0) \) denote the subclass of functions in \( T_M(A, B) \) satisfying \( f'(z_0) = -\frac{1}{z_0^2} \) (where \( 0 < z_0 < 1 \)).

In this chapter, we obtain sharp coefficient estimates for the class \( T_M(A, B, z_0) \). Radius of meromorphic convexity, integral transform of functions in \( T_M(A, B, z_0) \) have been studied. It is also shown that the class \( T_M(A, B, z_0) \) is closed under convex linear combination. In the last part, the convolution problem of these functions have been studied.
2. MAIN RESULTS

In this section we prove our main results.

**Theorem 2.1.** Let \( f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \) be regular in \( \mathbb{C} \) and belongs to \( T_M(A, B) \) if and only if

\[
\sum_{n=1}^{\infty} \{n (1 + B) + A + 1\} a_n \leq B - A. \tag{2.1}
\]

**Proof:** Consider the expression

\[
H(f, f') = |z f'(z) + f(z)| - |B z f'(z) + A f(z)|. \tag{2.2}
\]

Replacing \( f \) and \( f' \) by their series expansions we have, for \( 0 < |z| = r < 1, \)

\[
H(f, f') = \left| \sum_{n=1}^{\infty} (n + 1) a_n z^n \right| - \left| (A-B) \frac{1}{z} + \sum_{n=1}^{\infty} (A+bn) a_n z^n \right|
\leq \sum_{n=1}^{\infty} (n + 1) a_n r^n - (B-A) \frac{1}{r} + \sum_{n=1}^{\infty} (A+Bn) a_n r^n,
\]

or

\[
rH(f, f') \leq \sum_{n=1}^{\infty} \{n (1 + B) + A + 1\} a_n r^{n+1} - (B-A).
\]

Since this holds for all \( r, 0 < r < 1, \) making \( r \to 1, \) we have

\[
H(f, f') \leq \sum_{n=1}^{\infty} \{n (1 + B) + A + 1\} a_n - (B-A) \leq 0, \tag{2.3}
\]

in view of (2.1). From (2.2), we thus have

\[
\left| \frac{z f'(z)}{f(z)} + 1 \right| \leq 1.
\]

Hence \( f \in T_M(A, B). \)

Conversely, let \( f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \) and

\[
\left| \frac{z f'(z)}{f(z)} + 1 \right| \leq 1
\]
or
\[
\frac{\sum_{n=1}^{\infty} (n + 1) a_n z^n}{(A-B) - \sum_{n=1}^{\infty} (A + Bn) a_n z^n} \leq 1
\]

or

\[
\frac{\sum_{n=1}^{\infty} (n + 1) a_n z^{n+1}}{(B-A) - \sum_{n=1}^{\infty} (A + Bn) a_n z^{n+1}} \leq 1
\]

Since \( \text{Re} (z) \leq |z| \)

\[
\text{Re} \left\{ \frac{\sum_{n=1}^{\infty} (n + 1) a_n z^{n+1}}{(B-A) - \sum_{n=1}^{\infty} (A + Bn) a_n z^{n+1}} \right\} \leq 1
\]

choosing \( z = r \) with \( 0 < r < 1 \), we get

\[
\frac{\sum_{n=1}^{\infty} (n + 1) a_n r^{n+1}}{(B-A) - \sum_{n=1}^{\infty} (A + Bn) a_n r^{n+1}} \leq 1. \quad (2.4)
\]

Let \( S(r) = (B-A) - \sum_{n=1}^{\infty} (A + Bn) a_n r^{n+1} \).

\( S(r) \neq 0 \) for \( 0 < r < 1 \), \( S(r) > 0 \) for sufficiently small values of \( r \) and \( S(r) \) is continuous for \( 0 < r < 1 \). Hence \( S(r) \) can not be negative for any value of \( r \) such that \( 0 < r < 1 \). Upon clearing the denominator in (2.4) and letting \( r \to 1 \) we get

\[
\sum_{n=1}^{\infty} (n + 1) a_n \leq (B-A) - \sum_{n=1}^{\infty} (n + 1) a_n
\]

or

\[
\sum_{n=1}^{\infty} \{n (1 + B) + A + 1\} a_n \leq (B-A).
\]

Hence the Theorem.

**Theorem 2.2.** Let \( f(z) = \frac{a}{z} + \sum_{n=1}^{\infty} a_n z^n \). If \( f \) is regular in
and satisfies \( f'(z) = -\frac{1}{z^2} \), then \( f \in T_M(A, B, z_0) \) if and only if

\[
\sum_{n=1}^{\infty} \left\{ \frac{1}{n} (1 + B) + A + 1 \right\} - n (B-A) z_0^{n+1} \right\} a_n \leq (B-A), \quad a_n \geq 0. \tag{2.5}
\]

The result is sharp.

**Proof:** From Theorem 4.2.1, we know that a function \( g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \) regular in \( U' \) satisfies

\[
\left| \frac{z \frac{g'(z)}{g(z)} + 1}{B z \frac{g'(z)}{g(z)} + A} \right| < 1, \quad z \in U,
\]

if and only if

\[
\sum_{n=1}^{\infty} \left\{ \frac{1}{n} (1 + B) + A + 1 \right\} b_n \leq (B-A).
\]

Applying that result to the function \( g(z) = f(z)/a \), we find that \( f \) satisfies (1.1) if and only if

\[
\sum_{n=1}^{\infty} \left\{ \frac{1}{n} (1 + B) + A + 1 \right\} a_n \leq (B-A) a. \tag{2.6}
\]

Since \( f'(z_0) = -\frac{1}{z_0^2} \), we also have from the representation of \( f(z) \) that

\[
a = 1 + \sum_{n=1}^{\infty} n a_n z_0^{n+1}.
\]

Putting this value of \( a \) in the inequality (2.6), we have the required result.

For attaining the equality in (2.5), we choose the function

\[
f(z) = \frac{\left\{ \frac{1}{n} (1 + B) + 1 + A \right\} \frac{1}{z} + (B-A) z^n}{\left\{ \frac{1}{n} (1 + B) + 1 + A \right\} - n (B-A) z_0^{n+1}}. \tag{2.7}
\]

From (2.7), we have
\[ a_n = \frac{B - A}{\{n (1 + B) + 1 + A\} - n (B - A) \ z_0^n}, \]

or

\[ \left\{ n (1 + B) + 1 + A\right\} - n (B - A) \ z_0^n \right] a_n = (B - A), \]

and

\[ a = 1 + \sum_{n=1}^{\infty} n \ a_n \ z_0^n \]

\[ = 1 + \frac{n (B - A) \ z_0^n}{\{n (1 + B) + 1 + A\} - n (B - A) \ z_0^n} \]

\[ = \frac{\{n (1 + B) + 1 + A\}}{\{n (1 + B) + 1 + A\} - \{n (B - A) \ z_0^n\}} > 1. \]

**Theorem 2.3.** If \( f \in T_M(A, B, z_0) \), then \( f \) is meromorphically convex of order \( \delta \) \((0 \leq \delta < 1)\) in the disc \(|z| < R\), where

\[ R = \text{Inf.} \ n > 1 \ \left[ \frac{(1-\delta) \{n (1 + B) + 1 + A\}}{n (n + 2-\delta) (B - A)} \right]^{1/(n+1)} \]

This result is sharp for each \( n \) for functions of the form (2.7).

**Proof:** In order to establish the required result, it suffices to show that

\[ \left| 2 + \frac{z f''(z)}{f'(z)} \right| \leq 1 - \delta \]

or

\[ \left| \frac{f'(z) + [zf'(z)]'}{f'(z)} \right| \leq 1 - \delta \]

and

\[ \left| \frac{f'(z) + [zf'(z)]'}{f'(z)} \right| = \frac{\sum_{n=1}^{\infty} \frac{n (n + 1)}{a} a_n \ |z|^{n+1}}{1 - \sum_{n=1}^{\infty} \frac{n}{a} a_n \ |z|^{n+1}}. \]

This will be bounded by \((1-\delta)\) if

\[ \sum_{n=1}^{\infty} n (n + 2-\delta) \ a_n \ |z|^{n+1} \leq a (1-\delta). \]
Since \( a = 1 + \sum_{n=1}^{\infty} a_n z_0^{n+1} \), the above inequality can be written as

\[
\sum_{n=1}^{\infty} \frac{n \left( (n + 2 - \delta) \left| z \right|^{n+1} - (1 - \delta) z_0^n \right)}{(1 - \delta)} a_n \leq 1. \tag{2.8}
\]

Also by Theorem 2.2, we have

\[
\sum_{n=1}^{\infty} \frac{\left\{ n (1 + B) + 1 + A \right\} - n (B - A) z_0^{n+1}}{(B - A)} a_n \leq 1.
\]

Hence (2.8) will be satisfied if

\[
\frac{n \left( (n + 2 - \delta) \left| z \right|^{n+1} - (1 - \delta) z_0^n \right)}{(1 - \delta)} \leq \frac{\left\{ n (1 + B) + 1 + A \right\} - n (B - A) z_0^{n+1}}{(B - A)}, \quad \text{for each } n = 1, 2, \ldots
\]

\[
\left| z \right| < \left[ \frac{(1 - \delta) \left\{ n (1 + B) + 1 + A \right\}}{n (n + 2 - \delta) (B - A)} \right]^{1/(n + 1)},
\]

for each \( n = 1, 2, \ldots \).

This completes the proof of theorem. Sharpness follows if we take the same extremal function for which Theorem 2.2 is sharp.

**Theorem 2.4.** If \( f \in T_M(A, B, z_0) \), then the integral transform

\[
F(z) = c \int_{0}^{1} u^c f(uz) \, du, \quad \text{for } 0 < c < \infty,
\]

is in \( T_M(A', B', z_0) \), where

\[
\frac{1 + B'}{B' - A'} \leq \frac{(A + B + 2) (c + 2) + (B - A) c}{2c (B - A)} - \frac{z_0^2}{c}.
\]

The result is sharp for the function

\[
f(z) = \frac{(A + B + 2) \frac{1}{z} + (B - A) z}{(A + B + 2) - (B - A) z_0^2}.
\]
Proof: Let \( f(z) = \frac{a}{z} + \sum_{n=1}^{\infty} a_n z^n \in T_M(A, B, z_0) \),

then

\[
F(z) = c \int_0^1 u^c \left[ \frac{a}{uz} + \sum_{n=1}^{\infty} a_n (u^nz^n) \right] du
\]

\[
= c \int_0^1 \left[ u^{c-1} \cdot \frac{a}{z} + \sum_{n=1}^{\infty} a_n (u^{n+c}z^n) \right] du
\]

\[
= c \left[ \frac{u^c}{c} \cdot \frac{a}{z} + \sum_{n=1}^{\infty} a_n \frac{u^{n+c+1}}{(n+c+1)} z^n \right]_0^1
\]

\[
= \frac{a}{z} + \sum_{n=1}^{\infty} \frac{c}{n+c+1} a_n z^n.
\]

It is sufficient to show that

\[
\sum_{n=1}^{\infty} \frac{\{ n(1+B') + 1 + A' \} - n(B'-A') z_0^{n+1}}{(B'-A')(n+c+1)} c a_n \leq 1. \tag{2.9}
\]

Since \( f \in T_M(A, B, z_0) \) implies that

\[
\sum_{n=1}^{\infty} \frac{\{ n(1+B) + 1 + A \} - n(B-A) z_0^{n+1}}{(B-A)(n+c+1)} a_n \leq 1,
\]

(2.9) will be satisfied if

\[
\frac{\{ n(1+B') + 1 + A' \} - n(B'-A') z_0^{n+1}}{(B'-A')(n+c+1)} c e
\]

\[
\leq \frac{\{ n(1+B) + 1 + A \} - n(B-A) z_0^{n+1}}{(B-A)(n+c+1)}
\]

for each \( n, \)

\[
\frac{n(1+B') + 1 + A'}{(B'-A')} \leq \frac{\{ n(1+B) + 1 + A \} (n+c+1)}{c(B-A)} \]

\[
- \frac{n(n+1)}{c} z_0^{n+1}
\]

or
\[
\frac{1 + B'}{B' - A'} \leq \frac{n(1 + B) + 1 + A}{n + 1} \frac{(n + c + 1) + (B - A) c}{(n + 1)(B - A)c} - \frac{n}{c} z_0^{n+1}.
\]

The right hand side of (2.10) is an increasing function of \(n\), therefore putting \(n = 1\) in (2.10) we get:

\[
\frac{1 + B'}{B' - A'} \leq \frac{(A + B + 2)(c + 2) + (B - A) c}{2c (B - A)} - \frac{z_0^2}{c}.
\]

Hence the theorem.

**Theorem 2.5.** Let \(\gamma\) be a real number such that \(\gamma > 1\). If \(f \in T_M(A, B, z_0)\), then the function \(F\) defined by

\[
F(z) = \frac{(\gamma - 1)}{z^\gamma} \int_0^z t^{\gamma-1} f(t) \, dt
\]

also belongs to \(T_M(A, B, z_0)\).

**Proof:** Let \(f(z) = \frac{a}{z} + \sum_{n=1}^{\infty} a_n z^n\). Then from the representation of \(F(z)\), it follows that

\[
F(z) = \frac{a}{z} + \sum_{n=1}^{\infty} b_n z^n,
\]

where

\[
b_n := \frac{\gamma - 1}{\gamma + n} a_n.
\]

Therefore

\[
\sum_{n=1}^{\infty} \left[\frac{n(1 + B) + 1 + A}{n + 1} - n(B - A) z_0^{n+1}\right] b_n
\]

\[
= \sum_{n=1}^{\infty} \left[\frac{\gamma - 1}{\gamma + n}\right] \left[\frac{n(1 + B) + 1 + A}{n + 1} - n(B - A) z_0^{n+1}\right] a_n
\]

\[
\leq \sum_{n=1}^{\infty} \left[\frac{n(1 + B) + 1 + A}{n + 1} - n(B - A) z_0^{n+1}\right] a_n
\]

\[
\leq (B - A), \text{ by Theorem 2.2.}
\]

Hence \(F(z) \in T_M(A, B, z_0)\).
Theorem 2.6 Let \( f(z) = \frac{1}{z} \) and

\[
{f}_b(z) = \frac{n \{1 + B\} + 1 + A}{1 - n (B - A) z_0^{n+1}},
\]

\[ n = 1, 2, 3, \ldots \]

Then \( h \in T_M(A, B, z_0) \) if and only if it can be expressed in the form

\[
h(z) = \lambda f(z) + \sum_{n=1}^{\infty} \lambda_n f_n(z),
\]

where

\[
\lambda \geq 0 \quad \text{and} \quad \lambda + \sum_{n=1}^{\infty} \lambda_n = 1.
\]

Proof: Let us suppose that

\[
h(z) = \lambda f(z) + \sum_{n=1}^{\infty} \lambda_n f_n(z)
\]

\[
= \frac{a}{z} + \sum_{n=1}^{\infty} a_n z^n,
\]

where

\[
a = \lambda + \sum_{n=1}^{\infty} \frac{n \{1 + B\} + 1 + A\}}{1 - n (B - A) z_0^{n+1}} \lambda_n
\]

and

\[
a_n = \frac{(B-A) \lambda_n}{\{n \{1 + B\} + 1 + A\} - n (B - A) z_0^{n+1}}.
\]

Then, it is easy to see that \( f'(z_0) = - \frac{1}{z_0^2} \) and the condition (2.5) is satisfied. Hence \( h \in T_M(A, B, z_0) \).

Conversely let \( h \in T_M(A, B, z_0) \), and

\[
h(z) = \frac{a}{z} + \sum_{n=1}^{\infty} a_n z^n.
\]

Then, from (2.5), it follows that
\[ a_n \leq \frac{(B-A)}{\{n(1+B) + 1 + A\} - n(B-A)z_0^{n+1}}, \]

\[(n = 1, 2, 3, \ldots).\]

Setting

\[ \lambda_n = \left[ \frac{\{n(1+B) + 1 + A\} - n(B-A)z_0^{n+1}}{(B-A)} \right] a_n \]

and

\[ \lambda = 1 - \sum_{n=1}^{\infty} \lambda_n, \]

we have

\[ h(z) = \lambda f(z) + \sum_{n=1}^{\infty} \lambda_n f_n(z). \]

This completes the proof of theorem.

**Theorem 2.7.** Let \( f_j(z) = \frac{a_j}{z} + \sum_{n=1}^{\infty} a_{nj} z^n, \) \( j = 1, 2, \ldots, m. \)

If \( f_j \in T_M(A, B, z_0) \) for each \( j = 1, 2, \ldots, m, \) then the function \( h(z) \)

\[ = \frac{b}{z} + \sum_{n=1}^{\infty} b_n z^n \]

also belongs to \( T_M(A, B, z_0) \) where

\[ b = \sum_{j=1}^{m} \lambda_j a_j, \quad b_n = \sum_{j=1}^{m} \lambda_j a_{nj}, \quad (n = 1, 2, \ldots, m), \]

\[ \lambda_j \geq 0 \quad \text{and} \quad \sum_{j=1}^{m} \lambda_j = 1. \]

**Proof:** Since \( f_j \in T_M(A, B, z_0), \) then

\[ \sum_{n=1}^{\infty} \left[ \{n(1+B) + 1 + A\} - n(B-A)z_0^{n+1} \right] a_{nj} \leq (B-A) \]

\[ j = 1, 2, \ldots, m. \]

Therefore

\[ \sum_{n=1}^{\infty} \left[ \{n(1+B) + 1 + A\} - n(B-A)z_0^{n+1} \right] b_n \]
\[ \sum_{n=1}^{\infty} \left[ \{n (1 + B) + 1 + A\} - n (B-A) z_0^{n+1} \right] a_{n} \]

\[ \sum_{j=1}^{m} \lambda_{j} \sum_{n=1}^{\infty} \left[ \{n (1 + B) + 1 + A\} - n (B-A) z_0^{n+1} \right] a_{n} \]

\[ \leq \sum_{j=1}^{m} \lambda_{j} (B-A) = (B-A). \]

Hence by Theorem 2.2, \( h \in T_{M}(A, B, z_0) \).

**Theorem 2.8.** If \( f(z) = \frac{a}{z} + \sum_{n=1}^{\infty} a_{n} z^{n} \in T_{M}(A, B, z_0) \) and \( g(z) = \frac{b}{z} + \sum_{n=1}^{\infty} b_{n} z^{n} \) with \( b_{n} \leq 1 \) for \( n = 1, 2, \ldots \), then \( f \ast g \in T_{M}(A, B, z_0) \).

**Proof:** Let \( f(z) = \frac{a}{z} + \sum_{n=1}^{\infty} a_{n} z^{n} \) and \( g(z) = \frac{b}{z} + \sum_{n=1}^{\infty} b_{n} z^{n} \), then for convolution of functions \( f \) and \( g \) we can write

\[ \sum_{n=1}^{\infty} \left[ \{n (1 + B) + 1 + A\} - n (B-A) z_0^{n+1} \right] a_{n} b_{n} \]

\[ \leq \sum_{n=1}^{\infty} \left[ \{n (1 + B) + 1 + A\} - n (B-A) z_0^{n+1} \right] a_{n}, \]

because \( b_{n} \leq 1 \).

\[ \leq (B-A), \text{ by (2.5).} \]

Hence, by Theorem 2.2 \( f \ast g \in T_{M}(A, B, z_0) \).

**Note:** It will be of interest to find some other convolution results analogous to those of Juneja and Reddy [2].

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