RELATIONS BETWEEN THE SCALAR CURVATURES OF SUBMANIFOLDS WITH CONSTANT CURVATURE

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ABSTRACT

In this paper, the relations between the scalar curvatures of n-dimensional submanifold (hypersurface) \( N \), with zero curvature immersed in an \((n+1)\) dimensional submanifold \( \overline{N} \) with zero curvature in \( E^m \) \((m>n+1)\), have been investigated and some results have been obtained in terms of scalar, Gaussian and mean curvature of the submanifolds \( N \) and \( \overline{N} \).

INTRODUCTION

We shall assume throughout that all manifolds, maps, vector fields, etc. are differentiable of class \( C^\infty \).

Suppose that \( \overline{N} \) is an \((n+1)\)-dimensional submanifold of the Euclidean space \( E^m \) \((m>n+1)\), and \( N \) is an \( n \)-dimensional hypersurface immersed in an \((n+1)\)-dimensional submanifolds \( \overline{N} \) with constant curvature \( K \). Let \( p \) be a point of \( N \) and \( X^i \) the local coordinates around \( p \) in \( N \) such that \( X_i=\partial_i \) form an orthonormal basis of \( T_p \) \((N) \) at the point \( p \). \( \xi \) be orthonormal normal vector field of \( N \) in \( \overline{N} \), \( X \) and \( Y \) be two linear independent vectors at the point \( p \) and \( \gamma \) \((X,Y)\) be the plane section spanned by \( X \) and \( Y \). On the other hand, \( K(\gamma) \) is the constant for all plane sections \( \gamma \) in the tangent space \( T_p(N) \) at \( p \) where \( p \in N \), then \( N \) is a hypersurface with the constant curvature. The standard Riemann connection of \( E^m \) and Riemann connections of \( \overline{N} \) and \( N \) are denoted by

\[ \tilde{D}, \mathring{D} \text{ and } D, \text{ respectively.} \]

The Weingarten map \( L \) of \( N \) in \( \overline{N} \) is given by

\[ \mathring{D}_X \xi = L(X), \quad A \ X \in N_p \]

and \( \det L \) is the Gauss curvature at the point \( p \) of the hypersurface \( N \) of \( \overline{N} \).

(1.1)
Definition 1.1. Let \( M \) be an \( n \)-dimensional submanifold of the Euclidean space \( E^m \). Then
\[
\alpha : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M) \quad
\]
\[
(Y, Z) \to \alpha(Y, Z) = \sum_{j=1}^{m-n} \omega^j(Y, Z) \zeta_j
\]  
(1.2)
is called second fundamental form of \( M \). Where \( \omega^j \) denotes the coefficients of the second fundamental vector field in the direction of \( \zeta_j \), that is,
\[
\alpha(Y, Z) = \langle \alpha(Y, Z), \zeta_j \rangle.
\]  
[1]
To be \( Y, Z \in \mathcal{X}(N) \), Let \( \alpha_1(Y, Z) \) be the second fundamental form of \( \bar{N} \) in \( E^m \), then we have
\[
\overline{D}_Y Z = D_Y Z + \alpha_1(Y, Z)
\]  
(1.3)
and if \( \alpha_1(Y, Z) \) is the second fundamental form of \( N \) in \( E^m \), then we have
\[
\overline{D}_Y Z = D_Y Z + \alpha_2(Y, Z).
\]  
(1.4)
If \( Y \) and \( Z \) are vector fields of \( N \), then we have
\[
\overline{D}_Y Z = D_Y Z + \alpha_3(Y, Z).
\]  
(1.5)
Here (1.5) is the Gauss equation of \( N \) in \( \bar{N} \), where \( \alpha_3(Y, Z) \) is the second fundamental form \( N \) in \( \bar{N} \).

If we consider (1.5) and
\[
\alpha_3(Y, Z) = -\langle L(Y), Z \rangle, \zeta
\]  
(1.6)
we obtain
\[
\overline{D}_Y Z = D_Y Z - \langle L(Y), Z \rangle, \zeta
\]  
(1.7)
and using (1.7) in (1.3) we have
\[
\overline{D}_Y Z = D_Y Z - \langle L(Y), Z \rangle, \zeta + \alpha_1(Y, Z).
\]  
(1.8)
Moreover, if we consider (1.4) and (1.8) then we have
\[
\alpha_2(Y, Z) = -\langle L(Y), Z \rangle, \zeta + \alpha_1(Y, Z).
\]  
(1.9)
Let \( X \) and \( Y \) be orthonormal vectors at a point \( p \) and \( \gamma(X, Y) \) be the plane spanned by \( X \) and \( Y \). The sectional curvature \( K(\gamma) \) for \( \gamma(X, Y) \) is defined by
\[
K(\gamma) = K(X, Y, X, Y)
\]
or
\[
K(\gamma) = \langle X, R(X, Y) \rangle, Y \rangle
\]
where \( R \) is the curvature tensor.
It is easy to see that $K(\gamma)$ is independent of the choice of an orthonormal basis. So, we may give the following definition.

**Definition 1.2.** If $K(\gamma)$ is a constant for all plane in the tangent space $T_p(M)$ at $p$ for all points $P \in M$, then $M$ is called a space of constant curvature [2].

Let $M$ be an $n$-dimensional manifold immersed in an $m$-dimensional Riemann manifold $N$ of constant curvature $K$, $p$ be a point of $M$ and $X^i$ the local coordinates around $p$ in $M$ such that $X_i = \partial_i$ form an orthonormal basis of $T_p(M)$ at $p$ and also $\zeta_X$ be the orthonormal normal vector field of $M$. If we substitute

$$\zeta(X_i, X_j) = \alpha X(X_i, X_j) \zeta_X = \alpha^i_{ij} \zeta_X$$

then, we have $\alpha^i_{ji} = \alpha^i_{ij}$. Let $\langle \alpha \rangle$ denote the length of the second fundamental form $\alpha$, that is

$$\langle \alpha, \alpha \rangle = \langle \alpha \rangle^2 = \alpha^i_{ji} \alpha^j_{ii},$$

where $\alpha^i_{ji} = g^{iis} \alpha^s_{js}$.

**Definition 1.3.** If $E_1, E_2, \ldots, E_n$ are local orthonormal vector fields, then

$$R(X,Y) = \sum_{i=1}^{n} g(K(E_i,X)Y, E_i)$$

$$= \sum_{i=1}^{n} k(E_i,Y, E_i, X)$$

defines a global tensor field $R$ of type $(0,2)$ with local components

$$K_{ji} = K_{ij} = g^{is} K_{jis}.$$

Moreover, from the tensor field $R$ we can define a global scalar field

$$r = \sum_{i=1}^{n} R(E_i, E_i)$$

with local components

$$r = g^{ij} K_{ji}.$$

The tensor field $R$ and the function $r$ are called the Ricci tensor and scalar curvature.

From the Gauss equation, we find that the scalar curvature $r$ and the mean curvature vector $H$ satisfy the following relation.
\begin{align*}
r &= n^2 \|H\|^2 - <z>^2 + n(n-1)K. \quad [2] \\

\textbf{Theorem 1.1.} Let } r \text{ be the scalar curvature of } n\text{-dimensional submanifold } N \text{ with zero curvature and } \tilde{r} \text{ be the scalar curvature of } (n+1)\text{-dimensional submanifold } \tilde{N} \text{ with zero curvature in } E^m. \text{ Then, the relation between the scalar curvature of } N \text{ and the scalar curvature of } \tilde{N} \text{ is given by }

\begin{align*}
\tilde{r} - r &= (n+1)^2 \|H\|^2 - n^2 \|H\|^2 - 2 \sum_{i=1}^{n} <x_1(e_i, \zeta), x_1(e_i, \zeta) > \\
&= <x_1(\zeta, \zeta), x_1(\zeta, \zeta)> - (H^0)^2,
\end{align*}

\text{in } E^m, \text{ where } (H^0)^2 = \sum_{i=1}^{n} \lambda_i^2 \text{ and } \lambda_i = <L(e_i), e_i>.

\textbf{Proof:} By the hypothesis, we have

\[ S_p \{e_1, e_2, \ldots, e_n, e_{n+1} = \zeta \} = \chi(N) \]

and

\[ S_p \{e_1, e_2, \ldots, e_n \} = \chi(N). \]

Furthermore, since } K=0 \text{ for the scalar curvature of } M \text{ at the point } p \in M, \text{ by hypothesis from the following equation

\[ r = n^2 \|H\|^2 - <z>^2 + n(n-1)K, \]

we have

\[ r = n^2 \|H\|^2 - <z>^2. \quad (1.10) \]

If we consider (1.9), we have

\[ z_2(e_i, e_j) = -<L(e_i), e_j> + z_1(e_i, e_j). \]

From (1.2), it follows that

\[ <z_2>^2 = \sum_{i=1}^{n} <z_1(e_i, e_j), z_1(e_i, e_j) >. \]

Thus,

\[ <z_2>^2 = \sum_{i=1}^{n} <z_1(e_i, e_j), z_1(e_i, e_j) > + \sum_{i=1}^{n} \lambda_i^2, \text{ where } \lambda_i = <L(e_i), e_i>. \quad (1.11) \]

In the same way, from (1.2), we have
\[<\alpha_1>^2 = \sum_{i,j=1}^{n+1} <\alpha_1(e_i,e_j), \alpha_1(e_i,e_j)> \]
or
\[<\alpha_1>^2 = \sum_{i=1}^{n} <\alpha_1(e_i,e_i),\alpha_1(e_i,e_i)> + 2 \sum_{i=1}^{n} <\alpha_1(e_i,\zeta),\alpha_1(e_i,\zeta)> + <\alpha_1(\zeta,\zeta),\alpha_1(\zeta,\zeta)> \]  
(1.12)
since \(N\) and \(\overline{N}\) are manifolds with zero curvature in \(E^m\) and using the equation (1.10), (1.11) and (1.12) we obtain
\[\hat{r} - r = (n+1)^2 \|\mathbf{H}\|^2 - n^2 \|\mathbf{H}\|^2 - 2 \sum_{i=1}^{n} <\alpha_1(e_i,\zeta),\alpha_1(e_i,\zeta)> - <\alpha_1(\zeta,\zeta),\alpha_1(\zeta,\zeta)> - (H_0)^2. \]  
(1.13)
This completes the proof.

**Corollary 1.1.** If the scalar curvature of \(N\) is zero and if \(\zeta\) is asymptotic in \(\overline{N}\), then
\[\hat{r} = (n+1)^2 \|\mathbf{H}\|^2 - 2 \sum_{i=1}^{n} <\alpha_1(e_i,\zeta),\alpha_1(e_i,\zeta)> - (H_0)^2. \]

**Proof:** Since the scalar curvature of \(N\) is zero and \(\zeta\) is asymptotic in \(\overline{N}\) the proof is trivial by (1.10) and (1.13).

**Corollary 1.2.** If the scalar curvature of \(\overline{N}\) is zero, then
\[r = n^2 \|\mathbf{H}\|^2 + 2 \sum_{i=1}^{n} <\alpha_1(e_i,\zeta),\alpha_1(e_i,\zeta)> + (H_0)^2. \]

**Proof:** Since the scalar curvature of \(\overline{N}\) is zero, the proof is trivial by (1.10) and (1.13).

**Corollary 1.3.** Let \(p \in \overline{N}\). If \((e_i)_p\) and \(\zeta_p\) are conjugate two tangent vectors and if \(\zeta_p\) is asymptotic, then
\[\hat{r} - r = (n+1)^2 \|\mathbf{H}_p\|^2 - n^2 \|\mathbf{H}\|^2 - (H_0)^2. \]

**Proof:** Since, \((e_i)_p\) and \(\zeta_p\) are conjugate and \(\zeta_p\) is asymptotic, then the requirement results is obtained.

From definition 1.1 we write
\[ \alpha_1(e_i, e_i) = \sum_{k=1}^{m-n} x^k(e_i, e_i) \zeta_k. \]

For \( \zeta_k \in \mathcal{Z}(N) \), we have
\[ < \alpha_2(e_i, e_i), \zeta_k > = x^k(e_i, e_i) \]
or
\[ \alpha_2(e_i, e_i) = \sum_{k=1}^{m-n} < \alpha_2(e_i, e_i), \zeta_k > \zeta_k. \] (1.14)

Denoting the metric connection of the normal bundle \( N \) in \( E^m \) by \( D^\perp \), we write for \( e_1 \in \mathcal{Z}(N) \)
\[ \tilde{D} e_i \zeta_k = - A \zeta_k(e_i) + D^\perp e_i \zeta_k \]
or
\[ < \tilde{D} e_i \zeta_k, e_i > = < - A \zeta_k(e_i), e_i >. \]

Then we get
\[ < \alpha_2(e_i, e_i), \zeta_k > = < A \zeta_k(e_i), e_i >. \] (1.15)

Thus from (1.14) and (1.15) we have
\[ \alpha_2(e_i, e_i) = \sum_{k=1}^{m-n} < A \zeta_k(e_i), e_i > \zeta_k \] (1.16)

and
\[ \alpha_2(e_i, e_j) = \sum_{i=1}^{m-n} \sum_{j=1}^{m-n} < A \zeta_k(e_i), e_j > \zeta_1. \] (1.17)

Using (1.16) and (1.17) we write for \( k=1 \)
\[ \sum_{i'j=1}^{n} < \alpha_3(e_i, e_i), \alpha_2(e_j, e_j) > = \sum_{i'j=1}^{n} \sum_{k=1}^{m-n} < A \zeta_k(e_i), e_j > < A \zeta_k(e_i), e_j > \] (1.18)

considering that \( \sum_{i'=1}^{n} A \zeta_k(e_i) = \sum_{i'=1}^{n} a_{ij} e_j \) we get
\[ \sum_{i=1}^{n} < A \zeta_k(e_i), e_i > = \sum_{i'=1}^{n} < a_{ij} e_j, e_j > \]
or
\[ \sum_{i=1}^{n} < A \zeta_k(e_i), e_i > = \sum_{i=1}^{n} a_{ii}, i-j. \]

Hence we have obtained that
\[
\text{tr } A_{\zeta_k} = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \langle A_{\zeta_k}(e_i), e_i \rangle \quad (1.19)
\]

or

\[
\text{tr } A_{\zeta_k} = \sum_{j=1}^{n} a_{jj} = \sum_{j=1}^{n} \langle A_{\zeta_k}(e_j), e_j \rangle \quad (1.20)
\]

and that

\[
\sum_{k=1}^{m-n} (\text{tr } A_{\zeta_k})^2 = \sum_{i=1}^{n} \sum_{k=1}^{m-n} \langle A_{\zeta_k}(e_i), e_i \rangle \langle A_{\zeta_k}(e_j), e_j \rangle \quad (1.21)
\]

On the other hand we have

\[
\|H\| = \sum_{k=1}^{m-n} (\text{tr } A_{\zeta_k}/n)\zeta_k
\]

and so

\[
n^2 \|H\|^2 = \sum_{k=1}^{m-n} (\text{tr } A_{\zeta_k})^2. \quad (1.22)
\]

Then from (1.18), (1.21) and (1.22) we get

\[
\sum_{i,j=1}^{n} \langle \alpha_2(e_i, e_i), \alpha_2(e_j, e_j) \rangle = n^2 \|H\|^2.
\]

This gives for \(i = j,\)

\[
H = 1/n \sum_{i=1}^{n} \alpha(e_i, e_i).
\]

If \(H = 0\) at each point of \(N\) then \(N\) is minimal and so \(\alpha = 0.\) From (1.9), we write

\[
\alpha_1(e_i, e_i) = \langle L(e_i), e_i \rangle \zeta.
\]

Since the hypersurface \(N\) is totally geodesic, \(L = 0\) and so \(\alpha_1 = 0.\) Then from

\[
\bar{H} = 1/n+1 \sum_{i=1}^{n+1} \alpha_1(e_i, e_i) \text{ we have that } \bar{H} = 0, \text{ that is the submanifold } \bar{N} \text{ is minimal and also from } \alpha_1(\zeta, \zeta) = 0, \text{ we can say that } \zeta \text{ is an asymptotic direction in } \bar{N}. \text{ Therefore we have proved the assertion.}
Application 1.1. Let $\overline{N}_1$ be an 3-dimensional submanifold in $E^m$, given by the following parametric form

$$X = \{(a+k/\sqrt{2}) \cos u \cos v, (a+k/\sqrt{2}) \cos u \sin v, (a+k/\sqrt{2}) \sin u, k/\sqrt{2}, 0, \ldots, 0) | x_j = 0, j = 5, 6, \ldots, m, k \in \mathbb{R}\}$$

and let $S^2$ be a 2-hypersphere in $E^m$, given by the following parametric form

$$Y = \{(a \cos u \cos v, a \cos u \sin v, a \sin u, 0, \ldots, 0) | y_j = 0, j = 4, 5, \ldots, m, a > 0\}$$. If the scalar curvature of $S^2$ and $\overline{N}_1$ are, respectively, $r_b$ and $\bar{r}_a$ in $E^m$, then

$$r_a - r_b = 9 \|H_a\|^2 - 4 \|H_b\|^2 - \sin u/2(a+k/\sqrt{2})^2 + (H^0)^2.$$  

Indeed, we may write

$$y_1 = x_1 = e_1 = (-\sin u \cos v, -\sin u \sin v, \cos u, 0, \ldots, 0)$$

$$y_2 = x_2 = e_2 = (-\sin v, \cos v, 0, \ldots, 0)$$

$$y_3 = y_0 = (1/\sqrt{2} \cos u \cos v, 1/\sqrt{2} \cos u \sin v, 1/\sqrt{2} \sin u, -1/\sqrt{2}, 0, \ldots, 0)$$

then

$$\text{Sp} \left\{ e_1 | p, e_2 | p \right\} = T_{S^2}(p),$$

$$\text{Sp} \left\{ e_3 | p = \xi_0 | p, \partial_{\xi 1} | p, \partial_{\xi 2} | p, \ldots, \partial_{\xi m} | p \right\} = T^1_{S^2}(p)$$

and

$$\text{Sp} \left\{ e_1 | p, e_2 | p, e_3 | p = \xi_0 | p \right\} = T_{\overline{N}_1}(p),$$

$$\text{Sp} \left\{ \xi_1 | p, \partial_{\xi 1} | p, \ldots, \partial_{\xi m} | p \right\} = T^1_{\overline{N}_1}(p).$$

From (1.9) and (1.2) we have

$$\alpha_b(e_1, e_j) = -\langle L(e_1), e_j \rangle = \alpha_a(e_1, e_j),$$

$$\alpha_b(e_1, e_1) = \sum_{i=1}^2 <\alpha_b(e_1, e_1), \alpha_b(e_1, e_1)> + \sum_{i=1}^2 \lambda_i^2$$

(1.24)

and

$$\alpha_a(e_1, e_1) = \sum_{i=1}^2 <\alpha_a(e_1, e_1), \alpha_a(e_1, e_1)> + \sum_{i=1}^2 \lambda_i^2$$

(1.25)

Then, from (1.10), (1.24) and (1.25) we obtain
\[ r_a - r_b = 9 \| H_a \|^2 - 4 \| H_b \|^2 - 2 \sum_{i=1}^{2} < \alpha_a(e_i, \zeta_0), \alpha_a(e_i, \zeta_0) > - < \alpha_a(\zeta_0, \zeta_0), \alpha_a(\zeta_0, \zeta_0) > - (H_0)^2. \] (1.26)

If we put the values of \( e_1, e_2 \) and \( \zeta_0 \), given by (1.23), in (1.26) then we obtain

\[ r_a - r_b = 9 \| H_a \|^2 - 4 \| H_b \|^2 - \sin u / 2 \left( a + k \right) + (H_0)^2. \]

REFERENCES
