A FIXED POINT THEOREM ON BANACH SPACE AND ITS APPLICATIONS

H.K. PATHAK

Department of Mathematics Kalyan Mahavidyalaya Bhilai Nagar (M.P.), 49006 INDIA
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ABSTRACT

In Section 1 of this paper we have extended the results on fixed point of operators on Banach spaces of Bernfeld, Lakshmikantham and Reddy and Som for two set-valued mappings. In the proof of our theorems we have introduced a generalized version of a contractive type condition. In Section 2, we have investigated the solvability of certain non-linear functional equations in Banach spaces.

1. INTRODUCTION

Let X be a Banach space and let B(X) be the set of all non-empty, bounded subsets of X. Let C denotes the Banach space of all continuous functions from a finite closed interval [a, b] into X, such that

$$\|f\|_C = \sup_{a \leq t \leq b} \|f(t)\|_X \text{ for all } f \in C.$$  

The function \(\|A - B\|_{B(X)}\) with A,B in B(X) is defined by

$$\|A - B\|_{B(X)} = \sup \{ \|a - b\|_X : a \in A, b \in B \}.$$  

If A consists of a single point a we write

$$\|A - B\|_{B(X)} = \|a - B\|_{B(X)}$$

and if B also consists of a single point b we write

$$\|A - B\|_{B(X)} = \|a - b\|_{B(X)} = \|a - b\|_X.$$  

If follows easily from the definition that

$$\|A - B\|_{B(X)} = \|B - A\|_{B(X)} \geq 0$$

and

$$\|A - B\|_{B(X)} \leq \|A - C\|_{B(X)} + \|C - B\|_{B(X)},$$  

see Fisher [2,3] and Kaulgad and Pai [4].
Now, let $F$ be a mapping of $C$ into $B(X)$, a member $f \in C$ is said to be a fixed point of $F$ if $F(e)$ is in $Ff$ for some $e \in [a, b]$.

The intent of the present paper is to extend a result of Bernfeld et al. [4] for two set-valued mappings $F$ and $G$ of $C$ into $B(X)$, where $C$ is a Banach space of all continuous functions from a finite closed interval $[a, b]$ into $X$ and $B(X)$ being the set of all nonempty, bounded subsets of a Banach space $X$. One may also observed that we have introduced a generalized version of a contractive type condition.

Now, we give our main results as follows.

**Theorem 1.1.** Let $F : C \rightarrow B(X)$ and $G : C \rightarrow B(X)$ be two mappings which satisfy the following conditions for all $f, g \in C$, for a given $e \in [a, b]$

(a) $\|Ff - Gg\|_{B(X)} \leq \|f(e) - Gg\|_{B(X)}^{1-\gamma} \delta \{\|g(e) - Ff\|_{B(X)}\}^{\gamma} \{\|e - f\|_{C}\}^{\delta}$

where $0 \leq q \leq 1, \gamma, \delta \in [0, 1]$ with $0 < \gamma + \delta < 1$. Then (i) for a given $f_0 \in C$, every sequence $\{f_n : n=1,2,\ldots\}$ defined as $f_n(e)$ is in $Ff_{n-1} = X_n$ for the some $e \in [a, b]$ such that

$\|f_{n1} - f_{n2}\|_{C} = \|f_{n1}(e) - f_{n2}(e)\|_{X}$

for $n_1, n_2 = 0,1,2,\ldots$, converges to a point $f^*$ and $f^*$ is a common fixed point of $F$ and $G$. Further, $Ff^* = Gf^* = f^*(e)$ and $f^*(e)$ is the unique common point in $Ff^*$ and $Gf^*$, and

(ii) Let $\Omega f^* = \{f \in C : \|f - f^*\|_{C} = \|f^*(e)\|_{X}\}$,

where $f^*$ is a common fixed point of $F$ and $G$, then $F$ and $G$ have a unique common fixed point in $\Omega f^*$.

**Proof:** First we prove that $\{f_n\}$ is a Cauchy sequence. Put

$\alpha = \|f_0(e) - Gf_0\|_{B(X)}$

and suppose that the sequence $\{\|X_n - Gf_0\|_{B(X)} : n = 1,2,\ldots\}$ is unbounded. Then there exists an integer $n \geq 2$ such that

$\beta = \|X_n - Gf_0\|_{B(X)} \geq \|X_{n-1} - Gf_0\|_{B(X)}$

with $\beta > q^{1/(\gamma + \delta)} \alpha (1 - q^{1/(\gamma + \delta)})$ and so

$\|X_{n-r} - f_0(e)\|_{B(X)} \leq \|X_{n-r} - Gf_0\|_{B(X)} + \|Gf_0 - f_0(e)\|_{B(X)} \leq \beta + \alpha$

for $r = n-1, n$. But on using inequality (a), we have

$\beta = \|X_n - Gf_0\|_{B(X)} = \|Ff_{n-1} - Gf_0\|_{B(X)}$

$\leq q \{\|f_{n-1}(e) - Gf_0\|_{B(X)}\}^{1-\gamma} \delta \|f_0(e) - Ff_{n-1}\|_{B(X)}\}^{\gamma} \{\|f_{n-1} - f_0\|_{C}\}^{\delta}$
\[ \leq q \left\{ \| X_{n-1} - Gf_0 \|_{B(X)} \right\}^{1-\gamma} \delta \left\{ \| f_0(e) - X_n \|_{B(X)} \right\}^\gamma \left\{ \| f_{n-1}(e) - f_0(e) \|_{B(X)} \right\}^\delta \]
\[ \leq q \left\{ \| X_{n-1} - Gf_0 \|_{B(X)} \right\}^{1-\gamma} \delta \left\{ \| f_0(e) - X_n \|_{B(X)} \right\}^\gamma \left\{ \| X_{n-1} - f_0(e) \|_{B(X)} \right\}^\delta \]
\[ \leq q \beta^{1-\gamma \delta} (\beta + \varepsilon)^\gamma (\beta + \varepsilon)^\delta \]
which implies that
\[ \beta < q^{1/(1-\delta)} \varepsilon \left( 1 - q^{1/(1-\delta)} \right) \]
giving a contradiction. The sequence \( \{ X_n - Gf_0 \|_{B(X)} : n = 1, 2, \ldots \} \) must therefore be bounded.

Similarly, we can prove that the sequence \( \{ Fg_0 - Y_n \|_{B(X)} : n = 1, 2, \ldots \} \) is bounded. Since
\[ \| X_\gamma - Y_s \|_{B(X)} \leq \| X_\gamma - Gf_0 \|_{B(X)} + \| Gf_0 - Fg_0 \|_{B(X)} + \| Fg_0 - Y_s \|_{B(X)} \],
it follows that
\[ M = \sup \{ \| X_\gamma - Y_s \|_{B(X)} : r, s = 1, 2, \ldots \} \]
is finite. Now, for arbitrary \( \varepsilon > 0 \), choose a positive integer \( N \) such that
\[ q^N M < \varepsilon \]
Then for \( m, n \geq N \)
\[ \| X_m - Y_n \|_{B(X)} = \| Ff_{m-1} - Gg_{n-1} \|_{B(X)} \]
\[ \leq q \left\{ \| f_{m-1}(e) - Gg_{n-1} \|_{B(X)} \right\}^{1-\gamma} \delta \left\{ \| g_{n-1}(e) - Ff_{m-1} \|_{B(X)} \right\}^\gamma \left\{ \| f_{m-1} - g_{n-1} \|_{B(X)} \right\}^\delta \]
\[ \leq q \left\{ \| X_{m-1} - Y_n \|_{B(X)} \right\}^{1-\gamma} \delta \left\{ \| Y_{n-1} - X_m \|_{B(X)} \right\}^\gamma \left\{ \| f_{m-1}(e) - g_{n-1}(e) \|_{B(X)} \right\}^\delta \]
\[ \leq q \left\{ \| X_\gamma - Y_s \|_{B(X)} : m-1 \leq \gamma \leq m; n-1 \leq s \leq n \} \]
\[ \leq q^2 \{ \| X_\gamma - Y_s \|_{B(X)} : m-2 \leq \gamma \leq m; n-2 \leq s \leq n \} \]
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\[
\leq q^n \{ \| X_\gamma - Y_s \|_{B(X)} : m-N \leq \gamma \leq m; n-N \leq s \leq n \} \]
\[ \leq q^n M < \varepsilon. \]

Thus
\[ \| f_m - f_n \|_C = \| f_m(e) - f_n(e) \|_X \]
\[ \leq \| X_m - X_n \|_{B(X)} \]
\[ \leq \| X_m - Y_n \|_{B(X)} + \| Y_n - X_n \|_{B(X)} \leq 2 \varepsilon \]
for \( m, n \geq N \). The sequence \( \{ f_n : n = 1, 2, \ldots \} \) is therefore a Cauchy sequence in Banach space \( C \) and so has a limit \( f^* \). Further,
\[ \| f_m(e) - X_n \|_{B(X)} \leq \| X_m - X_n \|_{B(X)} \leq 2 \varepsilon \]

for \( m, n > N \) and so on letting \( m \) tends to infinity we have
\[ \| f^*(e) - X_n \|_{B(X)} \leq 2 \varepsilon \]

for \( n > N \). Thus
\[ \| f_n(e) - Gf^* \|_{B(X)} \leq \| Ff_{n-1} - Gf^* \|_{B(X)} \]
\[ \leq q \{ \| f_{n-1}(e) - Gf^* \|_{B(X)} \}^{1-\gamma-\delta} \{ \| f^*(e) - X_n \|_{B(X)} \}^\gamma \{ \| f_{n-1} - f^* \|_{C} \}^\delta \]
\[ \leq q \{ \| f_{n-1}(e) - Gf^* \|_{B(X)} \}^{1-\gamma-\delta(2\varepsilon)} \{ \| f_{n-1} - f^* \|_{C} \}^\delta \]

for \( n > N \). Letting \( n \) tends to infinity, we have
\[ \| f^*(e) - Gf^* \|_{B(X)} \leq q \{ \| f^*(e) - Gf^* \|_{B(X)} \}^{1-\gamma-\delta(2\varepsilon)} \{ \| f^* - f^* \|_{C} \}^\delta \]

which implies that
\[ \| f^*(e) - Gf^* \|_{B(X)} = 0 \]

and so
\[ Gf^* = \{ f^*(e) \} \]

We now have
\[ \| Ff^* - f^*(e) \|_{B(X)} \leq \| Ff^* - Gf^* \|_{B(X)} \]
\[ \leq q \{ \| f^*(e) - Gf^* \|_{B(X)} \}^{1-\gamma-\delta} \{ \| f^*(e) - Ff^* \|_{B(X)} \}^\gamma \{ \| f^* - f^* \|_{C} \}^\delta \]
\[ = 0. \]

It follows that \( Ff^* = \{ f^*(e) \} \) and \( f^* \) is a common fixed point of \( F \) and \( G \) and \( f^*(e) \) is a common point in \( Ff^* \) and \( Gf^* \).

Now, suppose that \( G \) has a second fixed point \( f^{**} \) so that \( f^{**}(e) \) is in \( Gf^{**} \). Then
\[ \| f^*(e) - f^{**}(e) \|_X \leq \| f^*(e) - Gf^{**} \|_{B(X)} \]
\[ \leq \| Ff^* - Gf^{**} \|_{B(X)} \]
\[ \leq q \{ \| f^*(e) - Gf^{**} \|_{B(X)} \}^{1-\gamma-\delta} \{ \| f^{**}(e) - Ff^* \|_{B(X)} \}^\gamma \{ \| f^* - f^{**} \|_{C} \}^\delta \]
\[ \leq q \{ \| Ff^* - Gf^{**} \|_{B(X)} \}^{1-\gamma-\delta} \{ \| Gf^{**} - Ff^* \|_{B(X)} \}^\gamma \{ \| Ff^* - Gf^{**} \|_{B(X)} \}^\delta \]

which implies
\[ \| Ff^* - Gf^{**} \|_{B(X)} \leq q \| Ff^* - Gf^{**} \|_{B(X)} \]

a contraction, since \( q < 1 \). It follows that
\[ \| Ff^* - Gf^{**} \|_{B(X)} = 0. \]

Thus, we have
\[ \| f^*(e) - Gf^{**} \|_{B(X)} = 0 \]
and then that $f^*(e) = f^*(e)$. The set $Gf^*$ is therefore singleton and $f^*(e)$ is in $Gf^*$. Similarly, the set $Ff^*$ is singleton and $f^*(e)$ is in $Ff^*$.

Finally, let $g^*(\neq f^*)$ be another common fixed point of $F$ and $G$ in $\Omega f^*$ where $f^*$ is a common fixed point of $F$ and $G$ in $\Omega f^*$, then

$$
\|f^*-g^*\|_C = \|f^*(e)-g^*(e)\|_{B(X)} = \|Ff^*-Gf^*\|_{B(X)} \\
\leq q \{\|f^*(e)-Gf^*\|_{B(X)}\}^{1-r} \{\|g^{**}(e)-Ff^*\|_{B(X)}\}^r \{\|f^*-g^*\|_C\}^\gamma \\
= q \{\|f^*(e)-g^*(e)\|_{B(X)}\}^{1-r} \{\|g^*(e)-f^*\|_{B(X)}\} \{\|f^*-g^*\|_C\}^\gamma \\
= q \|f^*-g^*\|_C \\
$$
a contradiction, since $q < 1$. It follows that

$$
\|f^*-g^*\|_C = 0.
$$

Hence $f^* = g^*$, that is, $F$ and $G$ have a unique common fixed point $f^*$ in $\Omega f^*$.

**Corollary 1.** Let $S: C \to X$ and $T: C \to X$ be two operators satisfying the following condition for all $f, g \in C$, and for a given $e \in [a,b]$

$$
\|Sf-Tg\|_X \leq q \{\|f(e)-Tg\|_X\}^{1-\gamma} \{\|g(e)-Sf\|_X\}^\gamma \{\|f-g\|_C\}^\delta
$$

where $0 \leq q \leq 1$, $\gamma, \delta \in [0,1]$ with $0 \leq \gamma + \delta \leq 1$. Then

(i) for a given $f_0 \in C$, every sequence $\{f_n : n=1,2,\ldots\}$ defined as $f_n(e) = Sf_{n-1}$ for the same $e$ such that

$$
\|f_{n1}-f_{n2}\|_C = \|f_{n1}(e)-f_{n2}(e)\|_X
$$

for $n_1, n_2 = 0,1,2,\ldots$, converges to a point $f^*$ and $f^*$ is a common fixed point of $S$ and $T$, and

(ii) Let $\Omega f^* = \{f \in C : \|f-f^*\|_C = \|f(e)-f^*(e)\|_X\}$, where $f^*$ is a common fixed point of $S$ and $T$, then $S$ and $T$ have a unique fixed point $f^*$ in $\Omega f^*$.

**Proof:** Define mappings $F$ and $G$ of $C$ into $B(X)$ by putting

$$
Ff = \{Sf\}, \ Gf = \{Tg\}
$$

for all $f, g \in C$. It follows that $F$ and $G$ satify the conditions of Theorem 1 and so there exists a point $f^*$ in $C$ with

$$
Ff^* = Gf^* = \{f^*(e)\}.
$$

Hence, $f^*$ is a common fixed point of $S$ and $T$. Further, it is easy to see that $f^*(e)$ is the unique point in $Ff^*$ and $Gf^*$. The uniqueness of $f^*$ is immediate.
The following corollary is an immediate consequence of Corollary 1.

**Corollary 2.** (Som [5]) Let \( T : C \rightarrow X \) be an operator which satisfies the following condition for all \( f, g \in C \) for a given \( e \in [a, b] \)

\[
\| Tf - Tg \|_X \leq q \left( \| f(e) - Tg \|_X \right)^{1-\gamma} \delta \left( \| g(e) - Tf \|_X \right)^{\gamma} \| f - g \|_C \delta
\]

where \( 0 \leq q < 1, \gamma, \delta \in [0, 1] \) with \( 0 \leq \gamma + \delta \leq 1 \). Then (i) for a given \( f_0 \in C \), every sequence \( \{ f_n = n \} \) defined as \( f_n(e) = Ff_{n-1} \) for the same \( e \) each that

\[
\| f_{n_1} - f_{n_2} \|_C = \| f_{n_1}(e) - f_{n_2}(e) \|_X
\]

for \( n_1, n_2 = 0, 1, 2, \ldots \), converges to a fixed point \( f^* \) of \( T \), and

(ii) Let \( \Omega f^* = \{ f \in C : \| f - f^* \|_C = \| f(e) - f^*(e) \|_X \} \), where \( f^* \) is a fixed point of \( T \), then \( T \) has a unique fixed point \( f^* \) in \( \Omega f^* \).

**Remark:** If we take \( \gamma = 0 \) and \( \delta = 1 \) in the above corollary, the results of Bernfeld et al. [1] follows.

2. **Application:** In this section we wish to investigate the solvability of certain non-linear functional equations in a Banach space.

**Theorem 2.1.** Let \( \{ f_n \} \) be a sequence of elements in \( C \), and let \( \{ g_n \} \) be a sequence of solutions to the equation \( \| Gf - Ff \|_{B(X)} = \| f_n(e) \|_X \) for the same \( e \in [a,b] \), \( n = 1, 2, \ldots \), where \( F \) and \( G \) are as in Theorem 1.1, \( f_n(e) \) is in \( Ff_{n-1} \) and \( g_n(e) \) is in \( Fg_{n-1} \). Then if \( \| f_n(e) \|_X \rightarrow 0 \) is \( n \rightarrow \infty \), the sequence \( \{ g_n \} \) converges to the unique solution of the equation \( Ff = Gf \).

**Proof:** First we observe that

\[
\| Fg_n - Gg_m \|_{B(X)} \leq q \left( \| g_n(e) - Gg_m \|_{B(X)} \right)^{1-\gamma} \delta \left( \| g_m(e) - Fg_n \|_{B(X)} \right)^{\gamma} \| g_n - g_m \|_C \delta
\]

\[
\leq q \left( \| Fg_n - Gf_m \|_{B(X)} \right)^{1-\gamma} \delta \left( \| Gg_m - Fg_n \|_{B(X)} \right)^{\gamma} \| g_n(e) - g_m(e) \|_{B(X)} \delta
\]

\[
\leq q \left( \| Fg_n - Gg_m \|_{B(X)} \right)^{1-\gamma} \delta \left( \| Fg_n - Gf_m \|_{B(X)} \right)^{\gamma} \| g_n - Fg_m \|_{B(X)} \delta
\]

\[
< q \left( \| Fg_n - Gg_m \|_{B(X)} \right)^{1-\gamma} \delta \left( \| Fg_n - Gg_m \|_{B(X)} \| Fg_n - Gg_m \|_{B(X)} \right)^{\gamma+\delta}
\]

which implies

\[
\| Fg_n - Gg_m \|_{B(X)} \leq q \left( \| f_m(e) \|_X \| Fg_n - Gg_m \|_{B(X)} \right)^{\gamma+\delta}
\]

So we have

\[
\| Fg_n - Gg_m \|_{B(X)} \leq \frac{q^{1/(\gamma+\delta)}}{1 - q^{1/(\gamma+\delta)}} \| f_m(e) \|_X
\]
Now, clearly
\[
\|g_n - g_m\|_c = \leq \|g_n(e) - g_m(e)\|_{B(X)} \leq \|Fg_n - Fg_m\|_{B(X)} \\
\leq \|Fg_n - Gg_m\|_{B(X)} + \|Fg_m - Gg_m\|_{B(X)} \\
\leq \frac{q^{1/(y+\delta)}}{1 - q^{1/(y+\delta)}} \|f_m(e)\|_X + \|f_m(e)\|_X.
\]

Hence \(\{g_n\}\) is a Cauchy sequence and so it will converge to some point, say \(g^*\). We further note from the inequality (a)
\[
\|g_n(e) - Gg^*\|_{B(X)} \leq \|Fg_n - Gg^*\|_{B(X)} \\
\leq q \left\{ \|g_n(e) - Gg\|_{B(X)} \right\}^{\gamma - \delta} \left\{ \|g^*(e) - Fg_n\|_{B(X)} \right\}^\gamma \left\{ \|g_n - g^*\|_c \right\}^\delta.
\]
Letting \(n \to \infty\), we have
\[
\|g^*(e) - Gg^*\| = 0
\]
and so
\[
Gg^* = g^*(e).
\]
We now have
\[
\|Fg^* - g^*(e)\|_{B(X)} \leq \|Fg^* - Gg^*\|_{B(X)} \\
\leq q \|g^*(e) - Gg^*\|_{B(X)}^{\gamma - \delta} \left\{ \|g^*(e) - Fg^*\|_{B(X)} \right\}^\gamma \left\{ \|g^* - g^*\|_c \right\}^\delta = 0.
\]
It follows that \(Fg^* = \{g^*(e)\}\) and \(\{g_n\}\) converges to the solution of the equation \(Ff = Gf\) as required. The rest of the proof is simple.

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REFERENCES


