ON THE PAIR OF AXOIDS UNDER THE SYMMETRIC HELICAL MOTION OF ORDER k IN THE EUCLIDEAN SPACE $E^n$

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ABSTRACT

The purpose of this paper, after giving a summary of known results about helical motion of order k and axoids in Euclidean space $E^n$, is to define the symmetric helical motion of order k and to obtain some results about integral invariants of the pair of axoids under the motion.

1. HELICAL MOTION OF ORDER k

A one parameter motion of a body in Euclidean space $E^n$ is generated by the transformation

$$\begin{bmatrix} \dot{x} \\ \dot{1} \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \dot{1} \end{bmatrix}, \quad AA^T = I_n$$

(1.1)

where $A: J \to SO(n)$ and $C: J \to IR^n$ are functions of differentiability class $C^r (r \geq 3)$ on real interval $J$. $\ddot{x}$ and $x$ correspond to the position vectors of the same point with respect to orthonormal coordinate systems of the moving space $E$ and fixed space $E$, respectively, [4].

The equation (1.1) by differentiation with respect to $t \in J$ yields

$$\dot{x} = B (x-C) + \dot{C}, \quad B = \dot{A}A^{-1}, \quad \ddot{x} = 0.$$ 

(1.2)

Since the matrix $A$ is orthogonal the matrix $B$ is skew. Therefore only in the case of even dimension it is possible that the determinant $|B|$ may not vanish. If $|B| \neq 0$ in $t \in J$, we get exactly one solution $Q(t)$ of the equation

$$B (Q-C) + \dot{C} = 0.$$ 

(1.3)

In this case, $Q$ is the center of the instantaneous rotation of the motion and called the pole of the motion in $t$. At the pole $Q$, the velocity

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vector vanishes by the equation (1.2). Therefore we get a differentiable curve α: I \rightarrow E of poles in the fixed space E, called the fixed pole curve. By (1.1) there is uniquely determined the moving pole \bar{\alpha}: J \rightarrow \overline{E} from the fixed pole curve, point to point. If |B| = 0, we obtain by the rules of Linear Algebra:

For every t \in J there exist a unit vector e(t) \in \ker B and \lambda(t) \in \mathbb{R} so that the solutions Q of the equation

\[ B \left( Q - C \right) + \dot{C} = \lambda e \]  

(1.4)

fill a uniquely determined linear subspace E_k(t) \subset \mathbb{R}^n with the dimension k = n - \text{rank} B. E_k(t) is the axis of the instantaneous screw (\lambda \neq 0) of the motion or the axis of the instantaneous rotation (\lambda = 0) and will be called the instantaneous axis of the motion in t \in J, [3]. In this second case, we obtain a generalized ruled surface of dimension k + 1 in \mathbb{R}^n generated by the instantaneous axis E_k(t), t \in J, which we call the fixed axoid \varnothing of the motion. The fixed axoid \varnothing determines the moving axoid \overline{\varnothing} in \overline{E} generator to generator by (1.1). The axoids \varnothing and \overline{\varnothing} touch each other along every common pair E_k(t) \subset \varnothing and \overline{E}_k(t) \subset \overline{\varnothing} for all t \in J by rolling and sliding upon each other under the motion, [5]. Such motion is called an (instantaneously) helical motion of order k in \mathbb{R}^n, [5].

2. GENERALIZED RULED SURFACES

In any k-dimensional generator E_k(t) of a (k + 1) dimensional generalized ruled surface (axoid, in [2] "(k + 1)-Regelflache") \varnothing \subset \mathbb{R}^n there exist a maximal linear subspace K_{k-m}(t) \subset E_k(t) of dimension k-m with the property that in every point of K_{k-m}(t) no tangent space of \varnothing is determined (K_{k-m}(t) contains all singularities of \varnothing in E_k(t)) or there exists a maximal linear subspace Z_{k-m}(t) \subset E_k(t) of dimension k-m with the property that in every point of Z_{k-m} the tangent space of \varnothing is orthogonal to the asymptotic bundle of the tangent spaces in the points of infinity of E_k(t) (all points of Z_{k-m}(t) have the same tangent space of \varnothing). We call K_{k-m}(t) the edge space in E_k(t) \subset \varnothing and Z_{k-m}(t) the central space in E_k(t) \subset \varnothing. A point of Z_{k-m}(t) is called a central point. If \varnothing possesses generators all of the same type the edge spaces resp. the central spaces generate a generalized ruled surface contained in \varnothing which call the edge ruled surface resp. the central ruled surface. For m = k the edge ruled surface degenerates in the edge of \varnothing, the central ruled surface in the line of striction. So the ruled surface
with edge ruled generalize the tangent surfaces of $E^3$, the ruled surface with central ruled surface generalize the ruled surfaces with line of striction of $E^3$.

For the analytical representation of a $(k + 1)$-dimensional ruled surface $\varnothing$ we choose a leading curve $a$ in the edge resp. central ruled surface $\Omega \subset \varnothing$ transversal to the generators. In [2] it is shown that there exists a distinguished moving orthonormal frame (ONF) of $\varnothing \{e_1, \ldots, e_k\}$ with the properties:

(i) $\{e_1, \ldots, e_k\}$ is an ONF of the $E_k(t) \subset \varnothing$,
(ii) $\{e_{m+1}, \ldots, e_k\}$ is an ONF of the $K_{k-m}(t)$ resp. $Z_{k-m}(t) \subset E_k(t)$,
(iii) $\dot{e}_i = \sum_{j=1}^{k} \alpha_{ij} e_j + K_i a_{k+1}, 1 \leq i \leq m$,

$$\dot{e}_{m+p} = \sum_{l=1}^{m} \alpha_{(m+k)l} e_1, \text{ with } K_1 > 0, \alpha_{ij} = -\alpha_{ji},$$

$$\alpha_{(m+k)(m+k)} = 0, 1 \leq p, x \leq k-m,$$
(iv) $\{e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m}\}$ is an ONF.

A moving ONF of $\varnothing$ with the properties (i)–(iv) is called a principal frame of $\varnothing$. If $K_1 > \ldots > K_k > 0$, the principal frame of $\varnothing$ is determined up to the signs. By a given principal frame the vectors $a_{k+1}, \ldots, a_{k+m}$ are well defined.

A leading curve $\alpha$ of $(k + 1)$-dimensional ruled surface $\varnothing$ is a leading curve of the edge resp. central surface $\Omega \subset \varnothing$ too iff its tangent vector has the form

$$\alpha(t) = \sum_{i=1}^{k} \zeta_i e_i + \eta_{m+1} a_{k+m+1},$$

(2.2)

where $\eta_{m+1} \neq 0$, $a_{k+m+1}$ is a unit vector well defined up to the sign with the property that $\{e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m}, a_{k+m+1}\}$ is an ONF of the tangential bundle of $\varnothing$. One shows: $\eta_{m+1}(t) = 0$, in $\mathfrak{t} \in \mathfrak{J}$ iff the generator $E_k(t) \subset \varnothing$ contains the edge space $K_{k-m}(t)$. If $\eta_{m+1}(t) \neq 0$, we call the $m$-magitudes

$$P_i = \frac{\eta_{m+1}}{K_i}, 1 \leq i \leq m$$

(2.3)
the principal parameters of distribution. These parameters are direct
generalizations of the parameter of distribution of the ruled surface
in \( \mathbb{E}^3 \) (see [2]). A \((k + 1)\)-dimensional ruled surface with central
ruled surface and no principal parameter of distribution \((m = 0)\) is a
\((k + 1)\)-dimensional cylinder.

Moreover the parameter of distribution of a generalized ruled
surface \( \varphi \) given in [3] by

\[
P = \frac{m}{\sqrt{|P_1 P_2 \cdots P_m|}}
\]  \hspace{1cm} (2.4)

and the total parameter of distribution of \( \varphi \) can be defined in [5] by

\[
D = \prod_{i=1}^{m} P_i.
\]  \hspace{1cm} (2.5)

Suppose that \( \varphi_i, \ 1 \leq i \leq k, \) are 2–dimensional closed principal
ruled surfaces such that the generators of \( \varphi_i \) have the direction of the
unit vectors \( e_i(t), \ 1 \leq i \leq k. \) Then, in the case \( m = k, \) there exist
\( k \)-pitches given by

\[
L_i = -\int_{0}^{P} \zeta_i(t) dt, \ 1 \leq i \leq k,
\]  \hspace{1cm} (2.6)

where \( p \in \mathbb{N} \) denotes a period of the motion.

Let \( \{e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m}, a_{k+m+1}\} \) be ONF of the tangential
bundle \( T(t) \) of \( \varphi. \) If we complete this ONF by an arbitrary
\( \{a_{k+m+2}, \ldots, a_n\} \) of the orthogonal complement, called a complementary
ONF. From the orthogonality conditions, then we obtain by differentiation, [3]:

\[
a_{k+1} = -K_1 e_1 + \sum_{j=1}^{m} \tau_{ij} a_{k+j} + w_i a_{k+m+1} + \sum_{\lambda=2}^{n-k-m} \tau_{i\lambda} a_{k+m+\lambda}, \ 1 \leq i \leq m.
\]  \hspace{1cm} (2.7)

Suppose that \( \text{dim } T(t) = k + m + 1. \) If \( \varphi \) is a closed ruled surface, the \( m \)-apex angles of \( \varphi \) can be define as

\[
\lambda_i = \int_{0}^{P} w_i(t) \ dt, \ 1 \leq i \leq m,
\]  \hspace{1cm} (2.8)

and also the apex angle of \( \varphi \) is defined, in [6], by

\[
\lambda = m\sqrt{|\lambda_1 \lambda_2 \cdots \lambda_m|}.
\]  \hspace{1cm} (2.9)
3. THE PAIR OF AXOIDS UNDER THE SYMMETRIC HELICAL MOTION

Let \( \overline{\alpha} \subset \mathcal{E} \) and \( \alpha \subset \mathcal{E} \) be moving and fixed pole curves of the helical motion of order \( k \). Suppose that \( \{ \vec{e}_1(t), ..., \vec{e}_k(t) \} \) is an ONF system at \( \overline{\alpha}(t) \) and let \( \mathcal{E}_k(t) = \text{Sp} \{ \vec{e}_1(t), ..., \vec{e}_k(t) \} \). Then \( \mathcal{E}_k(t) \) generates the moving axoid \( \overline{\mathcal{O}} \) with the leading curve \( \overline{\alpha} \) in \( \mathcal{E} \). A parametrization of \( \overline{\mathcal{O}} \) is

\[
\overline{\mathcal{O}}(t, \tilde{u}_1, ..., \tilde{u}_k) = \overline{\alpha}(t) + \sum_{i=1}^{k} \tilde{u}_i \vec{e}_i(t), \quad \tilde{u}_i \in \mathbb{R}, \quad t \in J. \tag{3.1}
\]

Let \( \{ e_1(t), ..., e_k(t) \} \) be an ONF system satisfying the following equation at the point \( \alpha(t) \) in the fixed space \( \mathcal{E} \):

\[
\mathcal{A} \vec{e}_i = -e_i, \quad 1 \leq i \leq k. \tag{3.2}
\]

\( \mathcal{E}_k(t) = \text{Sp} \{ e_1(t), ..., e_k(t) \} \) generates the fixed axoid \( \mathcal{O} \) with leading curve \( \alpha \) in \( \mathcal{E} \) by (1.1). And also a parametrization of \( \mathcal{O} \) is

\[
\mathcal{O}(t, u_1, ..., u_k) = \alpha(t) + \sum_{i=1}^{k} u_i e_i(t), \quad u_i \in \mathbb{R}, \quad t \in J. \tag{3.3}
\]

**Definition 3.1.** If a helical motion given by (1.1) satisfies the equation (3.2), then the motion is called a symmetric helical motion of order \( k \).

Let \( \overline{\mathcal{O}} \) and \( \mathcal{O} \) be \((k + 1)\)-dimensional moving and fixed axoids with the leading curves \( \overline{\alpha} \) and \( \alpha \), resp. (\( \overline{\alpha} \) and \( \alpha \) are the pole curves of the motion). Then we have the following equations, [1]:

\[
\dot{\overline{\alpha}} = \mathcal{A} \overline{\alpha}, \tag{3.4}
\]

\[
\dot{s} = \overline{\mathcal{A}}, \tag{3.5}
\]

where \( \overline{s} \) and \( s \) lengths of \( \overline{\alpha} \) and \( \alpha \), respectively. Then we have the following theorem.

**Theorem 3.2.** Under the symmetric helical motion of order \( k \) the moving and fixed axoids touch each other along every common pair \( \overline{\alpha} \) and \( \alpha \) for all \( t \in J \) by rolling and sliding upon each other.

Let \( \mathcal{E}_k(t) \) and \( \mathcal{E}_k(t) \) be the generator spaces of the axoids \( \overline{\mathcal{O}} \) and \( \mathcal{O} \), respectively. From (3.2) we have
\[ \dot{A} \hat{e}_i + A \hat{e}_i = -\hat{e}_i, \ 1 \leq i \leq k, \]
\[ -B \hat{e}_i + A \dot{\hat{e}}_i = -\dot{\hat{e}}_i, \]
\[ A \dot{\hat{e}}_i = -\dot{\hat{e}}_i, \ 1 \leq i \leq k, \ (B \hat{e}_i = 0). \]  
(3.6)

Then we immediately read off from (3.2) and (3.6).

**Theorem 3.3.** Under the symmetric helical motion of order \( k \), the generator spaces \( E_k(t) \) and \( E_k(t) \) correspond to each other by the equations (3.2) and (3.6).

Let \( \tilde{A}(t) \) and \( A(t) \) be the asymptotic bundles, with respect to the \( E_k(t) \) and \( E_k(t) \), of the axoids \( \overline{\varnothing} \) and \( \varnothing \) resp. Then \( \tilde{A}(t) \) and \( A(t) \) can be given resp. by

\[ \tilde{A}(t) = \text{Sp} \{ \tilde{e}_1, \ldots, \tilde{e}_k, \dot{\tilde{e}}_1, \ldots, \dot{\tilde{e}}_k \}, \]  
(3.7)

\[ A(t) = \text{Sp} \{ e_1, \ldots, e_k, \dot{e}_1, \ldots, \dot{e}_k \}. \]  
(3.8)

Suppose that \( \dim \tilde{A}(t) (= \dim A(t)) = k + m, \ 0 \leq m \leq k \), then \( m \) vectors of \( \tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_k \) are linearly independent. Let the linearly independent vectors are renumbered as \( \tilde{e}_{k+1}, \tilde{e}_{k+2}, \ldots, \tilde{e}_{k+m} \). Then the set

\[ \{ \tilde{e}_1, \ldots, \tilde{e}_k, \tilde{e}_{k+1}, \ldots, \tilde{e}_{k+m} \} \]  
(3.9)

is a basis of the asymptotic bundle \( \tilde{A}(t) \). Similarly, we get a basis for the asymptotic bundle \( A(t) \) as follows

\[ \{ e_1, \ldots, e_k, e_{k+1}, \ldots, e_{k+m} \}. \]  
(3.10)

By the Gram–Schmidt process form (3.9) and (3.10) we get the following orthogonal bases for \( \tilde{A}(t) \) and \( A(t) \) resp.,

\[ \{ \tilde{e}_1, \ldots, \tilde{e}_k, \tilde{y}_{k+1}, \ldots, \tilde{y}_{k+m} \}, \]  
(3.11)

\[ \{ e_1, \ldots, e_k, y_{k+1}, \ldots, y_{k+m} \}. \]  
(3.12)

Under the symmetric helical motion of order \( k \), the above orthogonal systems correspond to each other by the equation

\[ A \tilde{y}_{k+j} = -y_{k+j}, \ 1 \leq j \leq m. \]  
(3.13)

If we set

\[ \tilde{a}_{k+j} = \frac{\tilde{y}_{k+j}}{||\tilde{y}_{k+j}||}, \quad a_{k+j} = \frac{y_{k+j}}{||y_{k+j}||}, \ 1 \leq j \leq m, \]

then we get the following ONFs for \( \tilde{A}(t) \) and \( A(t) \) resp.,
\{\varepsilon_1, \ldots, \varepsilon_k, \tilde{a}_{k+1}, \ldots, \tilde{a}_{k+m}\} \quad (3.15)

\{e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m}\} \quad (3.16)

Therefore we have the following theorem.

\textbf{Theorem 3.4.} Under the symmetric helical motion of order \(k\), the asymptotic bundles \(\bar{A}(t)\) and \(A(t)\) correspond to each other by the following equations:

\[ A \tilde{e}_i = -e_i, \quad 1 \leq i \leq k, \]
\[ A \tilde{a}_{k+j} = -a_{k+j}, \quad 1 \leq j \leq m. \]  

(3.17)

Let \(\bar{T}(t)\) and \(T(t)\) be the tangential bundles of \(\bar{X}\) and \(X\) resp. If \(\dim \bar{T}(t) (= \dim T(t)) = k + m + 1\), then

\[ \{\varepsilon_1, \ldots, \varepsilon_k, \tilde{e}_{k+1}, \ldots, \tilde{e}_{k+m}, \tilde{a}\} \]  

is a basis \(\bar{T}(t)\) and

\[ \{e_1, \ldots, e_k, \tilde{e}_{k+1}, \ldots, \tilde{e}_{k+m}, a\} \]  

is a basis for \(T(t)\). Using the Gram–Schmidt process, we get following ONFs for \(\bar{T}(t)\) and \(T(t)\) resp.

\[ \{\varepsilon_1, \ldots, \varepsilon_k, \tilde{a}_{k+1}, \ldots, \tilde{a}_{k+m}, \tilde{a}_{k+m+1}\} \quad (3.20) \]
\[ \{e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m}, a_{k+m+1}\} \quad (3.21) \]

We can give the following theorem.

\textbf{Theorem 3.5.} Under the symmetric helical motion of order \(k\), the tangential bundles \(\bar{T}(t)\) and \(T(t)\) correspond to each other by the following equations:

\[ A \varepsilon_i = -e_i, \quad 1 \leq i \leq k, \]
\[ A \tilde{a}_{k+j} = -a_{k+j}, \quad 1 \leq j \leq m, \]
\[ A \tilde{a}_{k+m+1} = a_{k+m+1}. \]  

(3.22)

Now we can complete the ONF \(\{\varepsilon_1, \ldots, \varepsilon_k, \tilde{a}_{k+1}, \ldots, \tilde{a}_{k+m}, \tilde{a}_{k+m+1}\}\) of \(\bar{T}(t)\) to the ONF

\[ \{\varepsilon_1, \ldots, \varepsilon_k, \tilde{a}_{k+1}, \ldots, \tilde{a}_{k+m}, \tilde{a}_{k+m+1}, \ldots, \tilde{a}_n\} \]  

(3.23)

of \(E^n\). The orthonormal complement

\[ \{\tilde{a}_{k+m+2}, \ldots, \tilde{a}_n\} \]  

(3.24)
is called a complementary ONF of $\varnothing$.
If we set
\[
A\tilde{a}_{k+m+\lambda} = y_{k+m+\lambda}, \quad 2 \leq \lambda \leq n-k-m,
\]  
(3.25)
then we get an orthogonal complement \( \{y_{k+m+2}, \ldots, y_n\} \) of \( \varnothing \) under the symmetric helical motion of order \( k \). If we set
\[
a_{k+m+\lambda} = \frac{y_{k+m+\lambda}}{\|y_{k+m+\lambda}\|}, \quad 2 \leq \lambda \leq n-k-m,
\]  
(3.26)
then we have the following orthonormal complementary ONF of \( \varnothing \)
\[
\{a_{k+m+2}, \ldots, a_n\}.
\]  
(3.27)

**Theorem 3.6.** Under the symmetric helical motion of order \( k \), the complementary ONFs (3.24) and (3.27) satisfy the following equation:
\[
A\tilde{a}_{k+m+\lambda} = a_{k+m+\lambda}, \quad 2 \leq \lambda \leq n-k-m.
\]

Therefore, for the symmetric helical motion of order \( k \), we can give the following two corollaries:

**Corollary 3.7.** \( \mathbf{T}(t) \) and \( \mathbf{T}(t) \) being two tangential bundles which are 
 correspond to each other under the symmetric helical motion of order \( k \). Let \( \{\tilde{e}_1, \ldots, \tilde{e}_k, \tilde{a}_{k+m}, \tilde{a}_{k+m+1}, \ldots, \tilde{a}_n\} \) and 
\( \{e_1, \ldots, e_k, a_{k+m}, a_{k+m+1}, \ldots, a_n\} \) be two ONFs of \( E^n \) with respect to the \( \mathbf{T}(t) \) and \( \mathbf{T}(t) \) resp. Then we have the following equations: (3.2), (3.17), and
\[
A\tilde{a}_{k+m+\lambda} = a_{k+m+\lambda}, \quad 1 \leq \lambda \leq n-k-m.
\]  
(3.28)

**Corollary 3.8.** A symmetric helical motion of order \( k \) of \( E^n \) is a reflection with respect to the subspace \( \text{Sp}[\tilde{a}_{k+m+1}, \ldots, \tilde{a}_n] \) of dimension \( (n-k-m) \).

4. **THE INTEGRAL INVARIANTS OF THE PAIR OF AXOIDS WHICH CORRESPOND TO EACH OTHER UNDER THE SYMMETRIC HELICAL MOTION OF ORDER \( k \)**

**Theorem 4.1.** Let \( \varnothing \) and \( \varnothing \) be the \((k+1)-\) dimensional moving and fixed axoids which correspond to each other under the symmetric helical motion with the leading curves \( \tilde{z} \) and \( z \) resp., \( \{\tilde{e}_1, \ldots, \tilde{e}_k\} \) and 
\( \{e_1, \ldots, e_k\} \) being the principal ONFs of \( \varnothing \) and \( \varnothing \) resp., we have
\[
\tilde{\zeta}_i = -\zeta_i, \quad 1 \leq i \leq k,
\]  
(4.1)
\[ \eta_{m+1} = \eta_{m+1}, \]  \hspace{1cm} (4.2)

where \( \ddot{\tilde{z}} = \sum_{i=1}^{k} \ddot{\zeta}_i \ddot{e}_i + \ddot{\eta}_{m+1} \ddot{a}_{k+m+1} \) and \( \ddot{\alpha} = \sum_{i=1}^{k} \ddot{\zeta}_i \ddot{e}_i + \ddot{\eta}_{m+1} \ddot{a}_{k+m+1} \).

**Proof:**

\[ \ddot{A}(\ddot{\tilde{z}}) = \dot{A} \left( \sum_{i=1}^{k} \ddot{\zeta}_i \ddot{e}_i + \ddot{\eta}_{m+1} \ddot{a}_{k+m+1} \right), \]

\[ \ddot{A}(\ddot{\tilde{z}}) = \sum_{i=1}^{k} \ddot{\zeta}_i \dot{A}(\ddot{e}_i) + \ddot{\eta}_{m+1} \dot{A}(\ddot{a}_{k+m+1}). \]  \hspace{1cm} (4.3)

Using (3.2), (3.22), and (3.4) the theorem is proved.

**Theorem 4.2.** For

\[ \ddot{a}_{k+i} = -\ddot{K}_i \ddot{e}_i + \sum_{j=1}^{m} \ddot{\tau}_{ij} \ddot{a}_{k+j} + \ddot{w}_i \ddot{a}_{k+m+1} + \frac{\gamma_{k-m}}{\gamma_{k-m+1}} \sum_{\lambda=2}^{n} \ddot{\gamma}_{\lambda} \ddot{a}_{k+m+1}, \]

\[ 1 \leq i \leq m, \]  \hspace{1cm} (4.4)

\[ \ddot{a}_{k+i} = -\ddot{K}_i \ddot{e}_i + \sum_{j=1}^{m} \ddot{\tau}_{ij} \ddot{a}_{k+j} + \ddot{w}_i \ddot{a}_{k+m+1} + \frac{\gamma_{k-m}}{\gamma_{k-m+1}} \sum_{\lambda=2}^{n} \ddot{\gamma}_{\lambda} \ddot{a}_{k+m+1}, \]

\[ 1 \leq i \leq m, \]  \hspace{1cm} (4.4)

we have

\[ \ddot{K}_i = K_i, \quad \ddot{w}_i = -w_i, \quad \ddot{\gamma}_{i\lambda} = -\ddot{\gamma}_{\lambda i}, \quad 1 \leq i \leq m, \quad 2 \leq \lambda \leq n-k-m. \]  \hspace{1cm} (4.5)

**Proof:**

\[ \ddot{A}(\ddot{a}_{k+i}) = \dot{A} \left[ -\ddot{K}_i \ddot{e}_i + \sum_{j=1}^{m} \ddot{\tau}_{ij} \ddot{a}_{k+j} + \ddot{w}_i \ddot{a}_{k+m+1} + \frac{\gamma_{k-m}}{\gamma_{k-m+1}} \sum_{\lambda=2}^{n} \ddot{\gamma}_{\lambda} \ddot{a}_{k+m+1} \right]. \]

Since \( \dot{A} \) linear, using (3.2), (3.17), (3.22), and (3.28) we get.

\[ \ddot{A} \ddot{a}_{k+i} = \ddot{K}_i \ddot{e}_i - \sum_{j=1}^{m} \ddot{\tau}_{ij} \ddot{a}_{k+j} + \ddot{w}_i \ddot{a}_{k+m+1} + \frac{u-k-m}{\gamma_{k-m+1}} \sum_{\lambda=2}^{n} \ddot{\gamma}_{\lambda} \ddot{a}_{k+m+1}. \]  \hspace{1cm} (4.6)

From (3.17)

\[ \ddot{A} \ddot{a}_{k+i} + \ddot{A} \ddot{a}_{k+i} = -\ddot{a}_{k+i}, \]

\[ \ddot{A} \ddot{a}_{k+i} = -\ddot{a}_{k+i} - \ddot{A} \ddot{a}_{k+i}, \]

\[ \ddot{A} \ddot{a}_{k+i} = -\ddot{a}_{k+i} + \ddot{A} \ddot{a}_{k+i} \left( \ddot{a}_{k+i} = -\ddot{A}^{-1} \ddot{a}_{k+i} \right), \]
\[ a_{k+1} = -Aa_{k+1} + Ba_{k+1} (\hat{AA}^{-1} = B), \ 1 \leq i \leq m. \] (4.7)

If we set (4.7) in (4.4)', then we obtain

\[ A\hat{a}_{k+1} = K_1 e_1 - \sum_{j=1}^{m} \tau_{ij} a_{k+j} - w_j a_{k+m+1} + \sum_{\lambda=2}^{n-k-m} \gamma_{ij} a_{k+m+\lambda} + B a_{k+1}. \] (4.8)

Therefore from (4.6) and (4.8), the theorem is proved.

**Theorem 4.3.** If \( \hat{P}_1 \) and \( P_1 \) principal parameters of distribution of the axoids \( \overline{\varnothing} \) and \( \varnothing \) resp., then

\[ \hat{P}_1 = P_1, \ 1 \leq i \leq m. \] (4.9)

**Proof:** Using (4.2) and (4.5) in \( \hat{P}_1 = \overline{\gamma}_{m+1}/k_1 \), the theorem is proved.

**Corollary 4.4.** For the axoids \( \overline{\varnothing} \) and \( \varnothing \),

\[ \hat{P} = P, \] (4.10)
\[ \hat{D} = D. \] (4.11)

**Corollary 4.5.** Let \( L_1 \) and \( L_i \) be i-pitches of \( \overline{\varnothing} \) and \( \varnothing \) resp. under the closed symmetric helical motion of order \( k \). Then we have

\[ \overline{L}_i = -L_i, \ 1 \leq i \leq m = k, \] (4.12)
\[ \overline{L} = L, \] (4.13)

where \( \overline{L} = m^{\sqrt{\prod L_1 \cdots L_m}} \) (pitch of \( \overline{\varnothing} \)).

**Theorem 4.6.** Let \( \overline{\lambda}_1 \) and \( \lambda_i \) be i-apex angles of \( \overline{\varnothing} \) and \( \varnothing \) resp., under the closed symmetric helical motion of order \( k \). Then we have

\[ \overline{\lambda}_i = -\lambda_i, \ 1 \leq i \leq m = k. \] (4.14)

**Proof:** Since

\[ \overline{\lambda}_i = \int_0^P \overline{w}_i(t) \ dt \]

and \( \overline{w}_i = -w_i, \ 1 \leq i \leq m = k \), we get

\[ \overline{\lambda}_i = -\lambda_i, \ 1 \leq i \leq m = k. \]
Corollary 4.7. If $\overline{\lambda}$ and $\lambda$ are apex angles of the axoids $\mathcal{O}$ and $\mathcal{O}$ resp. under the symmetric helical motion of order $k$, then

$$\overline{\lambda} = \lambda.$$ 

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