THE FACTORIZATION OF ELEMENTS OF SO(n)
IN TERMS OF THE EULERS ANGLES

YUSUF YAYLI

Faculty of Sciences, Ankara University, Ankara.
(Received Nov. 6, 1991; Accepted July 14, 1992)

SUMMARY

In [1], the orthogonal matrices have been obtained with the aid of Skew-symmetric matrices in \( \mathbb{E}^2 \) and \( \mathbb{E}^3 \). In addition, an interpretation for these matrices has been given in this paper we tried to give the solution of this problem for \( \mathbb{E}^n \), \( n > 3 \).

It has been shown in [2] that the Lie algebra of the Lie group of orthogonal matrices \( O(n) \) consists of the skew symmetric matrices. Here, we explain the orthogonal \( n \times n \) matrices in terms of the exponential expansion of the bases of skew-symmetric matrices. Consequently we give a factorization of elements of SO(n) in terms of Euler Angles.

INTRODUCTION

Lie Groups And Lie Algebras

**Definition 1.1.** (Lie Groups). A Lie group is a group \( G \) which is, at the same time, a differentiable manifold such that the group operation

\[
\boxtimes : G \times G \rightarrow G
\]

\[
(a, b) \rightarrow ab^{-1}
\]

is a differentiable mapping of \( G \times G \) (product manifold) into \( G \).

**Definition 1.2.** (Lie Algebra). We define the Lie algebra of a Lie group \( G \) as the Lie algebra of the left invariant vector fields on \( G \). We have

\[
\mathfrak{g}(G) \cong T_G(e)
\]

where \( \mathfrak{g}(G) \) is the space of left invariant vector fields on \( G \) and \( e \) is the identity element of \( G \).

**Theorem 1.1.** The Lie algebra of the Lie group \( O(n) \) is the space of \( n \times n \) skew-symmetric matrices [2].
Proof. Let \( g \in O(n) \) then we have
\[
g^T g = e
\]
\[
d (g^T g) = 0
\]
\[
(g^{-1} dg)^T + g^{-1} dg = 0.
\]
Hence, \( \omega_{ij} g = g^{-1} i_k d g_k j \in T^*_{GL(n, \text{IR})}(g) \) we obtain
\[
[\omega_{ij} g]^T + [\omega_{ij} g] = 0
\]
or
\[
\omega_{ji} g \perp \omega_{ij} g = 0.
\]
For the inclusion mapping
\[
i^*: T^*_{GL(n, \text{IR})}(g) \rightarrow T^*_{O(n)}(g)
\]
we have
\[
i^* (\omega_{ji} g) + i^* (\omega_{ij} g) = 0
\]
\[
\xi_{ji} g + \xi_{ij} g = 0 \Rightarrow \xi_{ij} = - \xi_{ji}, i^* (\omega_{ji} g) = \xi_{ji} g,
\]
this proves the theorem.

If we denote the space of left invariant forms on \( O(n) \), by \( \Omega_L(O) \), then a base for this space is \( \{ \xi_{ij} \} \). Since a dual of this base is also a base for \( T_{O(n)} e \), then a base of \( T_{O(n)} e \) is
\[
\left\{ \frac{\partial}{\partial x_{ij}} - \frac{\partial}{\partial x_{ij}} , 1 \leq i, j \leq n \right\}
\]
\( T_{O(n)} (e) \), which is the Lie algebra of \( O(n) \), is the space of \( n \times n \) skew-symmetric matrices.

2. \( SO(n) \) And The Angles Of Euler

Theorem 2.1. If \( L \) is an \( n \times n \) skew-symmetric matrix then \( e^{L\theta} \in SO(n) \).

Proof. Let \( A = e^{L\theta} \)
\[
A A^T = (e^{L\theta})(e^{L\theta})^T
\]
\[
= e^{L\theta} e^{L\theta^T} \quad L^T = -L
\]
\[
= e^{L\theta-L\theta}
\]
\[
= e^0
\]
\[
= I_n.
\]
Since we have
\[ A^T A = I_n \]

then

\[ A \in O(n) \quad \ldots \quad (1) \]

Since we have \([3]\)

\[ \det e^L = e^{izL} \]

\[ L = [l_{ij}], \quad izL = 0 \]

\[ \det e^L = e^0 \]

\[ \det e^L = 1 \quad \ldots \quad (2) \]

So, (1) and (2) give us that \( A \in SO(n) \).

Hence we can say that the Lie algebra of \( O(n) \) is the space of skew-symmetric matrices. Let \( \{L_1, L_2, \ldots, L_{\frac{n(n-1)}{2}} \} \) be a base of this space, then the elements of this base can be written as

\[
L_1 = \begin{bmatrix}
0 & -1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

\[
L_2 = \begin{bmatrix}
0 & 0 & -1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

\[
L_{\frac{n(n-1)}{2}} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}
\]
Theorem 2.1. tells us that all the matrices $e^{L_1 \theta_1}$, $1 \leq i \leq \frac{n(n-1)}{2}$, are the rotation matrices. Each of these matrices represents a rotation about an axis. Hence we may consider here a composition of these $\frac{n(n-1)}{2}$ rotations.

$e^{L_1 \theta_1}$ causes a rotation about the axis $\frac{\partial}{\partial x_1}$ by the angle $\theta_1$.

$e^{L_2 \theta_2}$ causes a rotation about the axis $\frac{\partial}{\partial x_2}$ by the angle $\theta_2$.

$e^{\frac{L_n(n-1)}{2}} \frac{\theta_n(n-1)}{2}$ causes a rotation about the axis $\frac{\partial}{\partial x_{n(n-1)}}$ by the angle $\frac{\theta_n(n-1)}{2}$.

Further, if we have the product of these orthogonal matrices we obtain the matrix $A$ such as

$$A = e^{\frac{L_n(n-1)}{2}} \frac{\theta_n(n-1)}{2} \ldots e^{L_1 \theta_1}.$$  

Then $A$ is also an orthogonal matrix since the orthogonal matrices form a group under the matrix multiplication. Moreover, since

$$\det A = 1$$

we have that $A \in \text{SO}(n)$. When we consider the angles $\theta_i$ as Euler angles in (*), then the matrix $A \in \text{SO}(n)$ has a factorization in terms of the Euler angles.

3. An Example for the Case of $n = 3$

The Lie algebra of Lie group $O(3)$ consists of the matrices in the form

$$L = \begin{bmatrix}
-0 & -a & -b \\
 a & 0 & -c \\
 b & c & 0
\end{bmatrix}$$
each of which is the skew-symmetric so we can write this matrix as

\[
L = a \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.
\]

Denoting

\[
L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

we have

\[
e^{L_1 \theta_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & -\sin \theta_1 \\ 0 & \sin \theta_1 & \cos \theta_1 \end{bmatrix}, \quad e^{L_2 \theta_2} = \begin{bmatrix} \cos \theta_2 & 0 & \sin \theta_2 \\ 0 & 1 & 0 \\ -\sin \theta_2 & 0 & \cos \theta_2 \end{bmatrix},
\]

\[
e^{L_3 \theta_3} = \begin{bmatrix} \cos \theta_3 & -\sin \theta_3 & 0 \\ \sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

For these matrices,

\[e^{L_1 \theta_1}\] causes a rotation about \(\frac{\partial}{\partial x_1}\) by the angle \(\theta_1\),

\[e^{L_2 \theta_2}\] causes a rotation about \(\frac{\partial}{\partial x_2}\) by the angle \(\theta_2\),

\[e^{L_3 \theta_3}\] causes a rotation about \(\frac{\partial}{\partial x_3}\) by the angle \(\theta_3\).

In addition, if we have the product of these matrices we obtain

\[A = e^{L_3 \theta_3} \cdot e^{L_2 \theta_2} \cdot e^{L_1 \theta_1}\]

then we have that \(A \in \text{SO}(3)\).

REFERENCES

