ON THE CURVATURES OF THE PARALLEL HYPERSURFACES

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ABSTRACT

In this paper, we have shown that if \((n-2)\)-th mean curvature \(M_{n-2}\) of a hypersurface \(M\) is zero, then the sum of principal radii of curvatures of the parallel hypersurface \(M_r\) is constant. Secondly, we generalize a theorem of Bonnet which is for the parallel hypersurfaces in \(E^3\) to \(E^n\).

1. INTRODUCTION

In this section, we will give some fundamental definitions and theorems, which are necessary for the following sections.

Definition 1.1: Let \(M\) be an oriented hypersurface in \(E^n\). Define a map \(f\) as follows:

\[
f: M \rightarrow E^n
\]

\[
P \mapsto f(P) = P + rN_P,
\]

Where \(N\) is the unit normal vector field on \(M\), which gives the orientation of \(M\), and \(r\) is a given real number. Then \(M_r = f(M)\) is a hypersurface in \(E^n\) and furthermore \(M_r\) is called a parallel hypersurface to \(M\), in \(E^n\) \([4]\).

Definition 1.2: Let \(M\) be a hypersurface in \(E^n\). Let \(k_1, \ldots, k_{n-1}\) be the principal curvatures of \(M\). Put

\[
\begin{pmatrix} n-1 \\ s \end{pmatrix} M_s = \sum_{1 \leq i_1 < \cdots < i_s \leq n-1} k_{i_1} \cdots k_{i_s}, M_0 = 1,
\]

where \(\begin{pmatrix} n-1 \\ s \end{pmatrix} = \frac{(n-1)!}{(n-1-s)!s!}\).

We call \(M_s\) the \(s\)-th mean curvature of \(M\) \([1]\).
Theorem 1.1: Let $M$ and $M_r$ be parallel hypersurfaces in $E^n$. If $k$ is a principal curvature of $M$ at $P$, in the direction of $X$, then $k/(1 + rk)$ is the corresponding principal curvature of $M_r$ at $f(P)$ in the direction of $f^*(X)$, [4].

Theorem 1.2: Let $M$ and $M_r$ be parallel hypersurfaces in $E^n$. Then

$$H' = \sum_{i=1}^{n-1} \frac{k_i}{1 + rk_i}$$

and

$$K' = \prod_{i=1}^{n-1} \frac{k_i}{1 + rk_i}$$

where $k_i$, $1 \leq i \leq n-1$, denote the principal curvatures of $M$ and $H'$ and $K'$ stands for mean and Gaussian curvatures of $M_r$, respectively [2].

Theorem 1.3: Let $M$ and $M_r$ be parallel surfaces in $E^2$. If $M \subset E^3$, is a minimal surface ($H = 0$), then

$$\frac{1}{k_1'} + \frac{1}{k_2'} = 2r = \text{constant},$$

where $k_1'$ and $k_2'$ denote principal curvatures of $M_r$, [3].

The following Theorem due to Bonnet.

Theorem 1.4: (Bonnet): Let $M$ be a surface of constant positive Gauss curvature $K$ with no umbilics. Let $r_1 = \frac{1}{\sqrt{K}}$ and $r_2 = -\frac{1}{\sqrt{K}}$ define parallel sets $M_1$ and $M_2$, respectively.

Then,

i) $M_1$ and $M_2$ are immersions of $M$ which have constant mean curvatures $\sqrt{K}$ and $-\sqrt{K}$, respectively.

ii) If $M$ is a surface with constant mean curvature $H$ (non zero) and non-zero Gauss curvature, letting $r = -1/H$ yields a parallel set that is an immersion of $M$ with constant positive Gauss curvature $H^2$, [4].
2. GENERALIZATIONS OF THE THEOREM 1.3 AND THE THEOREM 1.4.

Theorem 2.1: Let M and M_r be parallel hypersurfaces in E^n. If (n-2)-th mean curvature M_{n-2} of M is zero, then

\[ \sum_{i=2}^{n-1} \frac{1}{k_r^i} = (n-1) r = \text{constant}, \]

where k_r^i, 1 \leq i \leq n-1, denote principal curvature of M_r at the point f(P).

Proof: From the Defition 1.2, (n-2)-th mean curvature M_{n-2} of M is

\[ \binom{n-1}{n-2} M_{n-2} = \sum_{1 \leq i_1 < \ldots < i_{n-2} \leq n-1} k_{i_1} \ldots k_{i_{n-2}} \]

\[ = \sum_{i=1}^{n-1} k_1 \ldots \hat{k}_i \ldots k_{n-1} \]

or

\[ (n-1) M_{n-2} = \sum_{i=1}^{n-1} k_1 \ldots \hat{k}_i \ldots k_{n-1}, \]

where the symbol \( \hat{\cdot} \) means that the term is omitted. On the otherhand, by the Theorem 1.1, we have

\[ k_r^i = \frac{k_i}{1 + rk_i}, 1 \leq i \leq n-1. \]

Now, we can show that,

\[ \sum_{i=1}^{n-1} \frac{1}{k_r^i} = \frac{\sum_{i=1}^{n-1} k_1 \ldots \hat{k}_i \ldots k_{n-1} + (n-1) r \prod_{i=1}^{n-1} k_i}{\prod_{i=1}^{n-1} k_i} . \]

Since,

\[ \sum_{i=1}^{n-1} k_1 \ldots \hat{k}_i \ldots k_{n-1} = 0 \]

thus, we get
\[ \sum_{i=1}^{n-1} \frac{1}{kr_i} = \frac{(n-1) \prod_{i=1}^{n-1} k_i}{\prod_{i=1}^{n-1} k_i} \]

\[ = (n-1) r = \text{constant}, \]

as desired.

Special case, \( n = 3 \): In this case, we find that,

\[ \sum_{i=1}^{2} \frac{1}{kr_i} = 2r = \text{constant}, \]

which is the same as Theorem 1.3.

**Theorem 2.2:** Let \( M \) and \( M_r \) be parallel hypersurfaces, in \( E^n \). Let \( M_i, 1 \leq i \leq n-1 \), \( i \)-th constant mean curvature of \( M \). If the following relation

\[ \sum_{i=2}^{n-1} \binom{n-1}{i} (i-1) r^i M_i = 1, \]

among the \( i \)-th mean curvatures of \( M \) holds then, the mean curvature \( H_r \) of the hypersurface \( M_r \) is equal to constant \( (1/r) \).

**Proof:** From the Theorem 1.2, we can write

\[ H_r = \sum_{i=1}^{n-1} \frac{k_i}{1 + rk_i}. \]

On the other hand, one can easily show that

\[ \sum_{i=1}^{n-1} \frac{k_i}{1 + rk_i} = \frac{1}{r} \left[ -1 + \sum_{s=2}^{n-1} r^s (s-1) \right] \sum_{1 \leq i_1 < \ldots < i_s \leq n-1} k_{i_1} \ldots k_{i_s} \]

\[ 1 + \sum_{s=2}^{n-1} \frac{r^s}{s} \sum_{1 \leq i_1 < \ldots < i_s \leq n-1} k_{i_1} \ldots k_{i_s} \]

Since,

\[ \binom{n-1}{s} M_s = \sum_{1 \leq i_1 < \ldots < i_s \leq n-1} k_{i_1} \ldots k_{i_s} \]

So we have the following,
\[ H^r = \frac{1}{r} \left[ \frac{-1 + \sum_{s=2}^{n-1} r^s (s-1) \binom{n-1}{s} M_s}{1 + \sum_{s=1}^{n-1} r^s \binom{n-1}{s} M_s} \right] \]

From the hypothesis
\[ \sum_{s=2}^{n-1} \binom{n-1}{s} (s-1) r^s M_s = 1, \]

So we get
\[ H^r = \frac{1}{r} \]

which completes the proof.

Special case, \( n = 3 \): In this case,
\[ \sum_{i=2}^{2} \binom{2}{i} (i-1) r^i M_i = 1 \]
or
\[ r^2 M_2 = 1 \]
From that, we obtain
\[ r = \pm \frac{1}{\sqrt{M_2}}. \]
Thus,
\[ H^r = \frac{1}{r} = \pm \sqrt{M_2} \]

On the other hand, from the definition of \( M_2 \) we know that
\[ M_2 = k_1 k_2 = K. \]
So we get
\[ H^r = \pm \sqrt{K} \]
that is to say;
If \( r = \frac{1}{\sqrt{K}} \), then \( H^r = \sqrt{K} \)

and

If \( r = -\frac{1}{\sqrt{K}} \), then \( H^r = -\sqrt{K} \).

This gives us the Theorem 1.4. (i).

**Theorem 2.3:** Let \( M \) and \( M_r \) be parallel hypersurfaces in \( E^n \). Denote \( M_i, 1 \leq i \leq n-1 \), for \( i \)-th constant mean curvatures of \( M \). If the following relation

\[
\sum_{i=1}^{n-2} \binom{n-1}{i} r^i M_i = -1, \]

among the \( i \)-th mean curvatures of \( M \) holds then, the Gaussian curvature \( K^r \) of the hypersurface \( M_r \) is equal to \( 1/r^{n-2} \).

**Proof:** From the Theorem 1.2,

\[
K^r = \prod_{i=1}^{n-1} \frac{k_i}{1 + r k_i} .
\]

On the other hand, we can calculate that

\[
\prod_{i=1}^{n-1} \frac{k_i}{1 + r k_i} = \frac{\prod_{i=1}^{n-1} k_i}{1 + \sum_{s=1}^{n-1} r^s \sum_{1 \leq i_1 < \ldots < i_s \leq n-1} k_{i_1} \ldots k_{i_s}} .
\]

Since,

\[
\binom{n-1}{s} M_s = \sum_{1 \leq i_1 < \ldots < i_s \leq n-1} k_{i_1} \ldots k_{i_s},
\]

So we can write that,

\[
K^r = \frac{M_{n-1}}{1 + \sum_{s=1}^{n-2} \binom{n-1}{s} r^s M_s + r^{n-1} M_{n-1}} .
\]

From the hypothesis,
\[ \sum_{i=1}^{n-2} \binom{n-1}{i} r^i M_1 = -1. \]

Thus,
\[ Kr = \frac{1}{r^{n-1}}. \]

So we obtain the desired equation.

Special Case, \( n = 3 \): In this case, from the hypothesis,
\[ 2rM_1 = -1 \]
or
\[ r = -\frac{1}{2M_1}. \]

On the otherhand, since
\[ 2M_1 = H \]
we have
\[ r = -\frac{1}{H}. \]

Thus, we find the following result,
\[ Kr = \frac{1}{r^2} = H^2 \]

which is the same as Theorem 1.4. (ii).

REFERENCES


