ON THE DEGREE OF APPROXIMATION OF A PERIODIC FUNCTION F BY ALMOST RIESZ - MEANS OF ITS CONJUGATE SERIES

By

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ABSTRACT

The present paper is concerned with the degree of approximation of certain functions belonging to the class Lip (\(\varphi(t)\), p) by almost Riesz means.

1. Let \(f\) be a 2\(\pi\)-periodic function integrable \(L^p\) (\(p > 1\)) and let

\[
f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

be its Fourier series.

The conjugate series of the Fourier series (1.1) is given by

\[
\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx)
\]

A function \(f \in \text{Lip} (\varphi(t), p) (p > 1)\) if

\[
\left\{ \int_0^{2\pi} |f(x + t) - f(x)|^p dt \right\}^{1/p} = O (\varphi(t))
\]

when \(\varphi(t)\) is a positive increasing function.

1. Definition (Lorentz [2]). A sequence \(\{S_n\}\) is said to be almost convergent to a limit \(S\),

\[
\text{if } \lim_{n \to \infty} \frac{1}{(n+1)} \sum_{k=p}^{n+p} S_k = S
\]

with respect to \(p\).
An almost convergence is a generalization of ordinary convergence.

2. Definition (Sharma and Qureshi [4]). A series \( \sum_{n=0}^{\infty} U_n \) with the sequence of partial sums \( \{S_n\} \) is said to be almost Riesz summable to \( S \), provided

\[
T_{n,p} = \frac{1}{p_n} \sum_{k=0}^{n} p_k S_{k+p} \rightarrow S \text{ as } n \to \infty
\]

uniformly with respect to \( p \), where

\[
S_{k,p} = \frac{1}{k+1} \sum_{\mu=1}^{k+p} S_{\mu}
\]

and \( \{p_n\} \) be a sequence of non-negative constants, such that \( p_0 > 0 \), \( P_n = p_0 + p_1 + \ldots + p_n \).

The Riesz means is regular if and only if \( P_n \to \infty \) with \( n \).

(see Theorem 1.4.4 of Peterson [3]).

Qureshi [1] proved the following theorem:

Theorem: The degree of approximation of a periodic function \( f(x) \), conjugate to a \( 2\pi \)-periodic function \( f(x) \) and belonging to the class \( \text{Lip } \alpha \), by almost Riesz means of its conjugate series, is given by

\[
\max_{0 \leq x \leq 2\pi} |f(x) - T_{n,p}(x)| = \begin{cases} 
O \left( \left( \frac{p_n}{P_n} \right)^{\alpha} \right) ; & 0 < \alpha < 1 \\
O \left( \frac{p_n}{P_n} \log \frac{p_n}{P_n} \right) ; & \alpha = 1
\end{cases}
\]

where, \( T_{n,p}(x) \) is the almost Riesz means of series (1.2) and Riesz means are regular such that \( 0 < p_n \uparrow \) with \( n \geq n_0 \). The object of this paper is to prove the following theorem.

Theorem: The degree of approximation of a periodic function \( \tilde{f}(x) \), conjugate to a \( 2\pi \)-periodic function \( f(x) \) and belonging to the class \( \text{Lip } (\tilde{\varphi}(t), p) \), \( (p > 1) \), by almost Riesz means of its conjugate series is given by

\[
\max_{0 \leq x \leq 2\pi} |f(x) - T_{n,p}(x)| = O \left( \tilde{\varphi} \left( \frac{p_n}{P_n} \right) \left( \frac{p_n}{P_n} \right)^{-1/p} \right)
\]
where $\tilde{T}_{n,p}(x)$ is the almost Riesz means of the series (1.2) and Riesz means are regular such that $0 < p_n \uparrow$ with $n > n_0$ where $\rho(t)$ is a positive increasing function and satisfies the following conditions:

\[
(i) \quad \left( \frac{p_n}{P_n} \right) \left( \frac{\rho(t)}{t^{1/p}} \right)^{1/p} dt = O \left( \frac{p_n}{P_n} \right) \\
(ii) \quad \left( \frac{\rho(t)}{t^{1/p+1}} \right)^{1/p} dt = O \left( \frac{p_n}{P_n} \right) \left( \frac{p_n}{P_n} \right)^{-1}
\]

Proof of the Theorem: Let $\tilde{S}_k$ be the k-th partial sum of the conjugate series (1.2). It is easy to show that:

\[
\tilde{S}_k - \tilde{f}(x) = \frac{1}{\pi} \int_0^\pi \frac{\cos \left( k + \frac{1}{2} \right) t}{2 \sin \frac{t}{2}} \psi(t) \, dt
\]

where $\psi(t) = f(x + t) - f(x - t)$

And $\tilde{S}_{k,p}(x) - \tilde{f}(x) = \frac{1}{k+1} \sum_{p=0}^{k+p} \{\tilde{S}_k(x) - \tilde{f}(x)\}$.

\[
= \frac{1}{\pi(k+1)} \int_0^\pi \psi(t) \frac{\sum_{k=0}^n \cos \left( k + \frac{1}{2} \right) t}{2 \sin \frac{t}{2}} dt
\]

\[
= \frac{1}{2\pi(k+1)} \int_0^\pi \psi(t) \frac{(\sin(pt) - \sin(k+p+1)t)}{2 \sin^2 \frac{t}{2}} dt
\]

We have

\[
\tilde{t}_{n,p}(t) - \tilde{f}(t) = \frac{1}{P_n} \sum_{k=0}^n p_k \{\tilde{S}_{k,p} - \tilde{f}(t)\}
\]

\[
= \frac{1}{2\pi P_n} \int_0^\pi \psi(t) \frac{\sum_{k=0}^n p_k [\sin(pt) - \sin(k+p+1)t]}{(k+1)2 \sin^2 \frac{t}{2}} dt
\]

Therefore
\[ \left| t_{n+p}(t) - \tilde{f}(t) \right| \leq \frac{1}{2\pi p_n} \int_0^\pi \left| \psi(t) \right| \sum_{k=0}^n \frac{p_k}{k+1} \]

\[ \cos(k+2p+1) \frac{t}{2} \sin(k+1) \cdot dt \]

\[ \sin^2 \frac{t}{2} \]

\[ = \frac{1}{2\pi p_n} \left[ \frac{p_n}{p_n} \int_0^\pi \left| \psi(t) \right| \cdot \left| \sum_{k=0}^n \frac{p_k}{k+1} \right| \right] \]

\[ \cos(k+2p+1) \frac{t}{2} \sin(k+1) \frac{t}{2} \]

\[ \sin^2 \frac{t}{2} \]

Now,

\[ I_1 = \frac{1}{2\pi p_n} \int_0^\pi \left| \psi(t) \right| \left| \sum_{k=0}^n \frac{p_k}{k+1} \right| \]

\[ \cos(k+2p+1) \frac{t}{2} \sin(k+1) \frac{t}{2} \]

\[ \sin^2 \frac{t}{2} \]

\[ = 0 \left[ \frac{1}{p_n} \int_0^\pi \left| \psi(t) \right| \left| \sum_{k=0}^n \frac{p_k}{k+1} \right| \right] \]

\[ \cos(k+2p+1) \frac{t}{2} \sin(k+1) \frac{t}{2} \]

\[ \sin^2 \frac{t}{2} \]
ON THE DEGREE OF APPROXIMATION...  

\[ \begin{align*} 
I_1 &= O \left[ \frac{1}{P_n} \left\{ \int_0^{p_n} | \psi(t)|^p \ dt \right\}^{1/p} \right] \\
&\quad \times \left\{ \frac{P_n}{P_n} \int_0^{p} \sum_{k=0}^{n} \frac{P_k}{k+1} \left( \cos (k+2p+1) \frac{t}{2} \sin (k+1) \frac{t}{2} \right) \left( \frac{1}{q} \right)^{1/q} \right\} \\
&\quad \times \left\{ \frac{P_n}{P_n} \int_0^{p} \frac{1}{t^{q/2}} \ dt \right\}^{1/q} \\
&\quad \times \left\{ \frac{P_n}{P_n} \int_0^{p} \left( \frac{\varphi(t)}{t^{1/p}} \right)^p \ dt \right\}^{1/p} \\
&\quad \times O \left( \frac{P_n}{P_n} \right)^{-1} + \frac{1}{q} \\
&= O \left( \varphi \left( \frac{p_n}{P_n} \right) \right) \left( \frac{p_n}{P_n} \right)^{-1} + \frac{1}{q} \\
&= O \left( \varphi \left( \frac{P_n}{P_n} \right) \left( \frac{p_n}{P_n} \right)^{-1} \right) \\
&\text{since } \frac{1}{p} + \frac{1}{q} = 1, \text{ such that } 1 \leq q \leq \infty, \\
\text{Similarly,} \\
I_2 &= O \left[ \frac{1}{P_n} \int_0^{\pi} | \psi(t)| \left\{ \frac{1}{P_n} \sum_{k=0}^{n} \frac{P_k}{k+1} \right\} \right] \\
\end{align*} \]
\[
\frac{\cos (k+2p+1) }{2} \int \frac{t}{2} \sin (k+1) \frac{t}{2} \sin^2 \frac{t}{2} \, dt
\]

\[
= O \left[ \frac{1}{P_n} \left\{ \int \frac{\varphi(t)}{t^{l/p+1}} \, dt \right\}^P \left\{ \int \frac{\varphi(t)}{t^{l/p+1}} \, dt \right\}^P \right] \left\{ \sum_{k=0}^{n} \frac{p_k}{(k+1)} \right\}
\]

\[
\times \frac{\sin \frac{t}{2}}{\sin^2 \frac{t}{2}} \left\{ \int q \, dt \right\}^q
\]

\[
= O \left[ \frac{1}{P_n} \left\{ \int \left( \frac{\varphi(t)}{t^{l/p+1}} \right) \, dt \right\}^P \right] \left\{ \sum_{k=0}^{n} \frac{p_k}{(k+2p+1)} \right\} \left\{ \int q \, dt \right\}^q
\]

\[
= O \left[ \frac{P_n}{P_n} \times \varphi \left( \frac{p_n}{P_n} \right) \left( \frac{P_n}{P_n} \right)^{-1} \left\{ \int \frac{1}{t^q} \, dt \right\}^q \right]
\]
ON THE DEGREE OF APPROXIMATION...

\[ = O \left[ \varphi \left( \frac{p_n}{P_n} \right) \left( \frac{p_n}{P_n} \right)^{-1+\frac{1}{q}} \right] \]

\[ = O \left[ \varphi \left( \frac{p_n}{P_n} \right) \left( \frac{p_n}{P_n} \right)^{-1} \right] \]

Since \{p_n\} is monotonic, increasing, we have

\[ \sum_{k=0}^{n} p_k \cos \left( k+2p+1 \right) \frac{t}{2} \leq p_n \sum_{k=0}^{n} \cos \left( k+2p+1 \right) \frac{t}{2} \]

\[ = O \left( \frac{p_n}{t} \right) \]

This completes the proof of the theorem.

REFERENCES


