SEPERATION PROPERTIES IN CATEGORIES OF
PREBORNOLOGICAL SPACES AND BORNOLOGICAL SPACES

MEHMET BARAN

Department of Mathematics, Faculty of Arts and Science, Erciyes University, Kayseri, Turkey

ABSTRACT

In this paper, an explicit characterization of each of the separation properties $T_0$, $T_1$, $PreT_2$, and $T_2$ is given in the topological categories of Prebornological Spaces and Bornological Spaces. Moreover, specific relationships that arise among the various $T_0$, $PreT_2$, and $T_2$ structures are examined in these categories.

1. INTRODUCTION

Let $E$ be a category and Sets be the category of sets.

1.1. Definition. A Functor $U: E \to \text{Sets}$ is said to be topological or $E$ is a topological category over Sets iff the following conditions hold:

1. $U$ is concrete i.e. faithful ($U$ is mono on hom sets) and amnestic (if $U(f) = \text{id}$ and $f$ is an isomorphism, then $f = \text{id}$).

2. $U$ has small fibers i.e. $U^{-1}(b)$ is a set for all $b$ in Sets.

3. For every $U$-source, i.e. family $g_i: b \to U(X_i)$ of maps in Sets, there exists a family $f_i: X \to X_i$ in $E$ such that $U(f_i) = g_i$ and if $U(h_i: Y \to X_i) = kg_i: UY \to b \to U(X_i)$, then there exists a lift $k: Y \to X$ of $k$: $UY \to UX$ i.e. $U(k) = k$. This latter condition means that every $U$-source has an initial lift. It is well known, see [3] p. 125 or [5] p. 278, that the existence of initial lifts of arbitrary $U$-source is equivalent to the existence of final lifts (the dual of the initial lifts) for arbitrary $U$-sink.

1.2. Definition. A Prebornological Space is a pair $(A, F)$ where $F$ is a family of subsets of $A$ that is closed under finite union and contains all finite nonempty subsets of $A$. See [4] p. 530. Furthermore, if $F \neq \emptyset$ and $F$ is hereditary closed, then $(A, F)$ is called a Bornological Space [4] p. 530 or [6] p. 1376. A morphism $(A, F) \to (A_1, F_1)$
of such spaces is a function \( f: A \to A_1 \) such that \( f(C) \in F_1 \) if \( C \in F \). We denote by P Born and Born respectively, the categories so formed and by P Born*, the full subcategory of P Born determined by those spaces \((A, F)\) with \( \emptyset \notin F \). [4] p. 530. The categories PBorn, P Born*, and Born are topological over sets. See [4] p. 530.

1.3. The discrete structure \((A, F)\) on \( A \) in \((P \text{ Born}, P \text{ Born}^*)\), Born is the set of all (nonempty) finite subsets of \( A \). See [4], p. 530.

1.4. A source \( \{f_i: (A, F) \to (A_i, F_i) \mid i \in I\} \) is initial in P Born, P Born*, and Born iff \( F = \{B / B \subseteq A, f_iB \in F_i \text{ for all } i\} \). See [4] p. 530.

1.5. An epi morphism \( f: (A_1, F_1) \to (A, F) \) is final in P Born or PBorn* (resp. Born) iff \( F = \{f(B) / B \in F_1\} \) (resp. \( F = \{B / B \subseteq A \text{ and } B \subseteq f(C) \text{ for some } C \in F_1\} \)).

An epi sink \( \{i_1, i_2: (A, F) \to (A_1, F_1)\} \) is final in P Born or PBorn* (resp. Born) iff \( F_1 = \{B / B \subseteq A_1 \text{ and } B \text{ is some finite union of sets of the form } i_k(C) \text{ with } C \in F, k = 1, 2\} \). See [4] p. 530.

1.6. Lemma, Suppose \( f: X \to Y \) is a morphism in P Born, PBorn*, or Born. If \( f \) has finite fibers i.e. \( f^{-1}(y) \) is a finite set for all \( y \) in \( Y \), then \( f \) reflects discreteness i.e. if \( Y \) is discrete, then so is \( X \).


Let \( X \) be a set and \( X^2 = X \times X \) be the cartesian product of \( X \) with itself. \( X^2 \Delta X^2 \) (two distinct copies of \( X^2 \) identified along the diagonal). A point \((x, y)\) in \( X^2 \Delta X^2 \) will be denoted by \((x, y)_1, ((x, y)_2) \) if \((x, y)\) is in the first (resp. second) component of \( X^2 \Delta X^2 \). Clearly \((x, y)_1 = (x, y)_2 \) iff \( x = y \). [2] p. 3.

1.7. Definitions. The principal axis map, \( A: X^2 \Delta X \to X^3 \) is given by \( A(x, y)_1 = (x, y, x) \) and \( A(x, y)_2 = (x, x, y) \). The skewed axis map, \( S: X^2 \Delta X \to X^3 \) is given by \( S(x, y)_1 = (x, y, y) \) and \( S(x, y)_2 = (x, x, y) \) and the fold map \( \nabla: X^2 \Delta X \to X^2 \) is given by \( \nabla(x, y)_1 = (x, y) \) for \( i = 1, 2 \).

Let \( U: E \to \text{Sets} \) be topological and \( X \) an object in \( E \) with \( UX = B \).

1.8. Definitions.

1. \( X \) is \( T_0 \) iff the initial lift of the \( U \)-source \( \{A: B^2 \Delta B^2 \to U (X^3) = B^3 \text{ and } \nabla: B^2 \Delta B^2 \to U D (B^2) = B^2\} \) is discrete.
2. X is $T'_0$ iff the initial lift of the U-source $\{id: B^2V_\triangle B^2 \rightarrow U (B^2V_\triangle B^2)' = B^2V_\triangle B^2 \text{ and } \triangledown: B^2V_\triangle B^2 \rightarrow U D (B^2) = B^2\}$ is discrete, where $(B^2V_\triangle B^2)'$ is the final lift of the U-sink $\{i_1, i_2: U (X^2) = B^2 \rightarrow B^2V_\triangle B^2\}$.

3. X is $T_1$ iff the initial lift of the U-source $\{S: B^2V_\triangle B^2 \rightarrow U (X^3) = B^3 \text{ and } \triangledown: B^2V_\triangle B^2 \rightarrow U D (B^2) = B^2\}$ is discrete.

4. X is $\text{PreT}_2$ iff the initial lift of the U-sources $A: B^2V_\triangle B^2 \rightarrow U (X^3) = B^3$ and $S: B^2V_\triangle B^2 \rightarrow U(X^3)$ agree.

5. X is $\text{PreT'}_2$ iff the initial lift of the U-source $S: B^2V_\triangle B^2 \rightarrow U (X^3)$ and the final lift of the U-sink $i_1, i_2: U (X^2) \rightarrow B^2V_\triangle B^2$ agree.

6. X is $\overline{T}_2$ iff X is $\overline{T}_0$ and $\text{PreT}_2$.

7. X is $T'_2$ iff X is $T'_0$ and $\text{PreT}'_2$.

8. X is $\Delta T_2$ iff the diagonal, $\Delta$, is closed in $X^2$. See [1] p. 8.

9. X is $ST_2$ iff the diagonal, $\Delta$, is strongly closed in $X^2$. See [1] p. 8.

1.9. Remark. We define $\pi_{ij}$ by $\pi_1 + \pi_1: B^2V_\triangle B^2 \rightarrow B$, where $\pi_1: B^2 \rightarrow B$ is the ith projection $i = 1, 2$. Note that $\pi_1 A = \pi_{11} = \pi_1 S$, $\pi_2 A = \pi_{21} = \pi_2 S$, $\pi_3 A = \pi_{12}$ and $\pi_3 S = \pi_{22}$. When showing that $A$ and $S$ are initial, it is sufficient to show that $(\pi_{11}, \pi_{21}$ and $\pi_{12})$, and $(\pi_{11}, \pi_{21}$ and $\pi_{22})$ are initial lifts, respectively. See [2] p. 13.

2. Separation Properties

In this section, we give explicit characterizations of the generalized separation properties for the topological categories of P Born, P Born*, and Born.

2.1. Lemma. If $\triangledown: (B^2V_\triangle B^2, K) \rightarrow (B^2, K_d)$ is in any one of P Born, P Born*, or Born, where $K_d$ is discrete structure on $B$, then $K$ is discrete.

Proof: This follows from 1.6 since the fibers of $\triangledown$ are finite.

2.2. Theorem. All objects in P Born, P Born* or Born are $T'_0$, $\overline{T}_0$, and $T_1$.

Proof: This follows from 2.1 and Definition 1.8.
2.3. Theorem. \( X = (B, F) \) in P Born or P Born* is Pre\( T_2 \) iff \( X \) is strictly hereditary closed i.e. if \( \varnothing \neq V \subset U \) and \( U \in F \), then \( V \in F \).

Proof: Suppose \( X \) is Pre\( T_2 \) i.e. by 1.4 and 1.9 for any subset \( W \) of the wedge if \( \pi_{11} W \in F \) and \( \pi_{21} W \in F \), then \( \pi_{12} W \in F \) iff \( \pi_{22} W \in F \). We must show that if \( \varnothing \neq V \subset U \), then \( V \in F \) if \( U \in F \). If \( V = U \), then clearly \( V \in F \). If \( \varnothing \neq V \neq U \) and \( V \subset U \), then let \( W = (V \times U) \setminus (U - V \times V) \) and note that \( \pi_{11} W = U \in F \), \( \pi_{21} W = U \in F \), \( \pi_{22} W = U \cup V = U \in F \), and \( \pi_{12} W = V \cup V = V \). Since \( X \) is Pre\( T_2 \), it follows that \( \pi_{12} W = V \in F \).

Conversely, we shall show that if \( X \) is strictly hereditary closed, then \( X \) is Pre\( T_2 \) i.e. if \( W = UVV \) is any subset of the wedge with \( \pi_{11} W = \pi_1 U \cup \pi_1 V \in F \), \( \pi_{21} W = \pi_2 U \cup \pi_1 V \in F \), then \( \pi_{12} W = \pi_1 U \cup \pi_2 W \in F \) iff \( \pi_{22} W = \pi_2 U \cup \pi_2 V \in F \). To this end, assume \( \pi_{11} W \) and \( \pi_{21} W \) are in \( F \). If \( U = \varnothing \neq V \), then \( \pi_{12} = \pi_2 V \in F \) iff \( \pi_{22} W = \pi_2 V \in F \). If \( U \neq \varnothing = V \), then \( \pi_{11} W = \pi_1 U \in F \), \( \pi_{21} W = \pi_2 U \in F \) and consequently \( \pi_{12} W = \pi_1 U \in F \) iff \( \pi_{22} W = \pi_2 U \in F \). If \( U = \varnothing = V \), then \( \pi_{11} W = \pi_1 U \cup \pi_1 V \in F \) and \( \pi_{21} W = \pi_2 U \cup \pi_1 V \in F \) imply by assumption that \( \pi_1 U \), \( \pi_2 U \), \( \pi_1 V \in F \) and consequently, \( \pi_{12} W = \pi_1 U \cup \pi_2 V \in F \) iff \( \pi_{22} W = \pi_2 U \cup \pi_2 V \in F \). If \( X \) is in P Born and \( U = \varnothing = V \), then \( W = \varnothing \) and if \( \pi_{11} W = \varnothing = \pi_{21} W \in F \), then \( \pi_{12} W = \varnothing \in F \) iff \( \pi_{22} W = \varnothing \in F \). This completes the proof.

2.4. Theorem. \( X = (B, F) \) in P Born or P Born* is Pre\( T'2 \) iff \( X \) is hereditary closed.

Proof: Suppose \( X \) is Pre\( T'2 \) i.e. by 1.4, 1.9, and 1.5 for any subset \( W \) of the wedge (a) \( \pi_{11} W \in F \), \( \pi_{21} W \in F \), and \( \pi_{22} W \in F \) iff (b) \( W = i_1 W_1 \cup i_2 W_2 \) for some \( W_1, W_2 \in F^2 \) where \( F^2 \) is defined by \( N \in F^2 \) iff \( \pi_1 N \in F \) and \( \pi_2 N \in F \). We will show that if \( U \in F \) and \( V \subset U \), then \( V \in F \). If \( V = U \), then \( V \in F \). If \( V \neq U \) and \( V \subset U \), then let \( W = V^2 V \setminus (U - V)^2 \) and clearly \( \pi_{11} W = U = \pi_{21} W = \pi_{22} W \in F \). Since \( X \) is Pre\( T'2 \), it follows that \( W = i_1 W_1 \cup i_2 W_2 \) and consequently \( W_1 = V^2 \in F^2 \). Thus, \( \pi_1 W_1 = V \in F \).

Conversely, suppose \( X \) is hereditary closed. We will show that \( X \) is Pre\( T'2 \) i.e. (a) and (b) are equivalent. To show (b) implies (a) note that if \( W = i_1 W_1 \cup i_2 W_2 \), then clearly \( \pi_{11} W = \pi_1 W_1 \cup \pi_1 W_2 \in F \), \( \pi_{21} W = \pi_2 W_1 \cup \pi_1 W_2 \in F \), and \( \pi_{22} W = \pi_2 W_1 \cup \pi_2 W_2 \in F \) (since \( W_1 \) and \( W_2 \) are in \( F^2 \) iff \( \pi_1 W_1 \in F \) and \( \pi_2 W_1 \in F \), and \( \pi_1 W_2 \)
\( \in F \) and \( \pi_2 W_2 \in F \). On the other hand, if \( W = UVV \), where \( U, V \) are subsets of \( B^2 \), and \( \pi_{11} W = \pi_1 U \cup \pi_1 V \in F \), \( \pi_{21} W = \pi_2 U \cup \pi_1 V \in F \), and \( \pi_{22} W = \pi_2 U \cup \pi_2 V \in F \), then, by assumption \( \pi_i U \) and \( \pi_i V \) are in \( F \) for all \( i = 1, 2 \). and consequently, \( U, V \) are in \( F^2 \). Clearly, \( W = i_1 UV \cup i_2 V \) and thus (a) implies (b). Therefore (a) and (b) are equivalent i.e. \( X \) is \( \text{Pre} \mathcal{T}'_2 \).

2.5. **Theorem.** \( X = (B, F) \) in \( \text{P Born} \) or \( \text{P Born}^* \) is \( \mathcal{T}'_2 \) iff \( X \) is strictly hereditary closed i.e. if \( \emptyset \neq V \subset U \) and \( U \in F \), then \( V \in F \).

**Proof:** Combine 2.2, 2.3, and Definition 1.8.

2.6. **Theorem.** \( X = (B, F) \) in \( \text{P Born} \) or \( \text{P Born}^* \) is \( \mathcal{T}'_2 \) iff \( X \) is hereditary closed.

**Proof:** Combine 2.2, 2.4, and Definition 1.8.

2.7. **Remark.** In \( \text{P Born} \) and \( \text{P Born}^* \), \( \text{Pre} \mathcal{T}'_2 \) and \( \mathcal{T}'_2 \) imply \( \text{Pre} \overline{\mathcal{T}}_2 \) and \( \overline{\mathcal{T}}_2 \), respectively.

2.8. **Theorem.** Every object in \( \text{Born} \) is \( \text{Pre} \overline{\mathcal{T}}_2 \), \( \text{Pre} \mathcal{T}'_2 \), \( \overline{\mathcal{T}}_2 \), and \( \mathcal{T}'_2 \).

**Proof:** This follows from the fact that \( X \) is hereditary closed.

2.9. **Theorem.** Let \( X = (B, F) \) be in \( \text{P Born} \), \( \text{P Born}^* \) or \( \text{Born} \). \( X \) is \( \Delta \mathcal{T}_2 \) iff \( B = \emptyset \) or a point.

**Proof:** [1] p. 17.

2.10. **Theorem.** All \( X \) in \( \text{P Born} \), \( \text{P Born}^* \), or \( \text{Born} \) are \( \text{ST}_2 \).

**Proof:** [1] p. 17.

2.11. **Remark.** Except for \( \Delta \mathcal{T}_2 \), all of the other separation properties defined in 1.8 are equivalent in \( \text{Born} \). Some of the \( \text{"T}_2\) structures could be equal while others could be different. For example, in \( \text{Born} \), \( \mathcal{T}'_2 \), \( \text{ST}_2 \) and \( \overline{\mathcal{T}}_2 \) are all equivalent and all are implied by but are different from \( \Delta \mathcal{T}_2 \). In \( \text{P Born} \) and \( \text{P Born}^* \), \( \mathcal{T}'_2 \) and \( \overline{\mathcal{T}}_2 \) are equivalent, are implied by \( \Delta \mathcal{T}_2 \), and imply \( \text{ST}_2 \).

**BornoLojik Ve Prebornolojik Kategorî uzaylarinda ayrılma aksiyonları**

**Özet**

Bu çalışmada, Bornolojik uzaylar ve Prebornolojik uzaylarında \( T_0, T_1, \text{Pre T}_2 \) ve \( T_2 \) ayrılıma özelliklerinin her birinin açık bir karakter-
rezisyonu verildi. Bundan başka, bu kategorilerde değişik $T_0$, Pre $T_2$ ve $T_2$ yapıları arasında ortaya çıkan özel ilişkiler incelendi.

REFERENCES


