ON THE CONGRUENCES OF LINES GENERATED BY THE INSTANTANEOUS SCREWING AXES CONNECTED WITH SOME SURFACES

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In this paper, the surfaces $\vec{z}$ and $\vec{z}$ in [2] have been considered as the base surface $\vec{x}$. And they are referred to their lines of curvature. The congruences generated by the instantaneous screwing axes of the moving trihedrons, moving along the lines of curvature on $\vec{z}$ and $\vec{z}$, have been investigated. And some corresponds with [1] have been explained.

1. INTRODUCTION

1. Let a surface $\vec{x}$ be referred to its lines of curvature. In case the congruences $\vec{y}$ and $\vec{y}$, generated by the instantaneous screwing axes $\vec{G}$ and $\vec{G}$ of the moving trihedrons connected with these lines are normal congruences, the surfaces which generate them are consecutively,

$$\vec{z} = r + \frac{1}{b} \vec{g}$$

(1.1)

$$\vec{r} = \vec{x} + \frac{1}{r} \vec{z} = \vec{z} + \frac{1}{b} \vec{n}, \vec{g} = \frac{\vec{r}x_2 + \vec{q} \vec{z}}{\sqrt{r^2 + q^2}}$$

and

$$\vec{z} = r + \frac{1}{\beta} \vec{g}$$

(1.2)

$$\vec{r} = \vec{x} + \frac{1}{r} \vec{z} = \vec{z} + \frac{1}{\beta} \vec{n}, \vec{g} = \frac{\vec{r}x_1 + \vec{q} \vec{z}}{\sqrt{i^2 + q^2}}$$
((1) and [2]).

2. The Surfaces $\mathbf{z}$, $\mathbf{z}$ as base surfaces.

1) The derivative formulas of the moving trihedron ($\mathbf{z}_2^\prime$, $\mathbf{z}_1^\prime$, $\mathbf{n}$) of the lines of curvature $u = \text{const.}$ with tangent $\mathbf{z}_2$ on the surface $\mathbf{z}$, may be stated as

$$\mathbf{z}_2^\prime = -a\mathbf{z}_1 + b\mathbf{n}$$

$$\mathbf{z}_1^\prime = a\mathbf{z}_2$$

$$\mathbf{n}_2 = -b\mathbf{z}_2$$

(2.1)

Here, $\tau_{g_1} = (\mathbf{z}_2 \dot{\times} \mathbf{n} \mathbf{n}_2^\prime) = 0$, $\mathbf{a} = (\mathbf{n} \mathbf{n}_2 \mathbf{z}_2^\prime)$, $\mathbf{b} = -(\mathbf{z}_2 \mathbf{n} \mathbf{n}_2^\prime)$.

And the derivative formulas of the moving trihedron ($\mathbf{z}_1^\prime$, $\mathbf{z}_2^\prime$, $\mathbf{n}$) of the lines of curvature $v = \text{const.}$ on the surface $\mathbf{z}$ are

$$\mathbf{z}_1^\prime = -a\mathbf{z}_2 + b\mathbf{n}$$

$$\mathbf{z}_2^\prime = a\mathbf{z}_1$$

$$\mathbf{n}_1 = -b\mathbf{z}_1$$

(2.1)

Here $\tau_{g_2} = (\mathbf{z}_1 \dot{\times} \mathbf{n} \mathbf{n}_1^\prime) = 0$, $-a = (\mathbf{n} \mathbf{z}_2 \mathbf{n}_2^\prime)$, $\mathbf{b} = -(\mathbf{z}_2 \mathbf{n} \mathbf{n}_1^\prime)$.

The surface $\mathbf{z}$ being considered as a base surface and during its motion on the lines of curvature $u = \text{const.}$, if the axes of the moving trihedron ($\mathbf{z}_2^\prime$, $\mathbf{z}_1^\prime$, $\mathbf{n}$) are denoted by $\mathbf{Z}_1 = \mathbf{z}_1 + \varepsilon \mathbf{z}_1^\prime$, $\mathbf{Z}_2 = \mathbf{z}_2 + \varepsilon \mathbf{z}_2^\prime$, $\mathbf{Z}_3 = \mathbf{n} + \varepsilon \mathbf{n}_0$, ($\varepsilon^2 = 0$), its instantaneous screwing axis $\mathbf{G}^*$ becomes,
\[ \tilde{G}^* = \frac{\varepsilon \tilde{Z}_2 + \tilde{b}\tilde{Z}_1 + \tilde{a}\tilde{Z}_3}{\sqrt{\tilde{a}^2 + \tilde{b}^2}}, \quad (\tilde{a} \neq 0, \tilde{b} \neq 0). \] 

(2.2)

Seperating \( \tilde{G}^* \) into its real and dual parts, we get

\[ \frac{\varepsilon}{\tilde{G}^*} = \frac{\tilde{b}z_1 - an}{\sqrt{\tilde{a}^2 + \tilde{b}^2}} + \varepsilon \frac{z_2 + bz_1 + an}{\sqrt{\tilde{a}^2 + \tilde{b}^2}} = \frac{\varepsilon}{\tilde{g}^*} + \frac{\varepsilon}{\tilde{g}_0}. \] 

(2.2)

\[ \frac{\varepsilon}{\tilde{g}^*} = \frac{\tilde{b}z_1 + an}{\sqrt{\tilde{a}^2 + \tilde{b}^2}} = \frac{z_2 + bz_1 + an}{\sqrt{\tilde{a}^2 + \tilde{b}^2}}. \]

From (2.1) and from the above definitions of \( \tilde{a}, \tilde{b} \) we find,

\[ \tilde{b} = \frac{1}{\lambda} \quad \left( \lambda = \frac{1}{\tilde{b}} \right) \]

(2.3)

\[ \tilde{a} = \frac{q}{\lambda r} \quad \left( \frac{\tilde{a}}{\tilde{b}} = \frac{q}{r} \right). \]

If we substitute these into \( \frac{\varepsilon}{\tilde{g}^*} \), we find

\[ \frac{\varepsilon}{\tilde{g}^*} = \tilde{\xi} \]

(2.4)

From this the following theorem may be stated:

2.1. Theorem

The instantaneous screwing axis \( \tilde{G}^* \) of the trihedron \( (\tilde{z}_2, -\tilde{z}_1, \tilde{n}) \), moving along the lines of curvature \( u = \text{const.} \) on the base surface \( \tilde{z} \), coincides with the axis \( \tilde{X}_3 \) carrying the surface normal \( \tilde{\xi} \) of the trihedron \( (\tilde{x}_1, \tilde{x}_2, \tilde{\xi}) \) moving along the lines of curvature \( v = \text{cons} \) on the base surface \( \tilde{x} \).

The following theorem may be derived from (2.3).
2.2. Theorem

During the motion considered in Theorem 2.1, the ratio of the geodesic curvatures and the normal curvatures of the surface $\vec{x}$ and $\vec{z}$, are equal
\[
\left( \frac{q}{r} = \frac{\alpha}{b} \right).
\]
In other words, the angles between the surface normals and the principal normals of these surfaces are equal ($\tan \theta = \tan \gamma$).

2) As the derivative formulas of the trihedron $\left( \vec{z}_-, \vec{z}_+, \vec{n}_+ \right)$ of the lines of curvature $v = \text{const.}$ On the surface $z$, we may write.

\[
\vec{z}_- = -\vec{x}z_+ + \beta \vec{n},
\]
\[
\vec{z}_+ = \alpha \vec{z}_-,
\]
\[
\vec{n}_+ = -\beta \vec{z}_-.
\]

(2.5)

Here, $\tau_{gu} = \left[ \vec{z}_- \ n_+ \ n_- \right]_1 = 0$, $\alpha = \left[ \vec{z}_- \vec{z}_+ \vec{z}_- \right]_1$, $\beta = \left[ \vec{z}_- \ n_- \right]_1$.

Also from (2.5) the derivative formulas of the trihedron $\left( \vec{z}_-, -\vec{z}_+, \vec{n}_+ \right)$ of the lines of curvature $u = \text{const.}$ on $z$, are

\[
\vec{z}_- = -\vec{x}z_+ + \beta \vec{n},
\]
\[
\vec{z}_+ = \alpha \vec{z}_-,
\]
\[
\vec{n}_+ = -\beta \vec{z}_-.
\]

(2.5)

Here, $\tau_{gv} = \left[ \vec{z}_- \ n_+ \ n_- \right]_2 = 0$, $\alpha = \left[ \vec{z}_- \vec{z}_+ \vec{z}_- \right]_2$, $\beta = \left[ \vec{z}_- \ n_- \right]_2$. 
The surface \( \tau \) being considered as a base surface and during its motion on the lines of curvature \( \nu = \text{const.} \), if the axes of the moving trihedron \( (z_1, z_2, \mathbf{n}) \) are denoted by \( \mathbf{Z}_1 = z_1 + \varepsilon z_2 + \frac{\varepsilon}{2} \mathbf{n}, \mathbf{Z}_2 = z_2 + \frac{\varepsilon}{10} \mathbf{n}, \mathbf{Z}_3 = \mathbf{n} + \varepsilon \mathbf{n}_0 \), its instantaneous screwing axis \( \mathbf{G}^{**} \) becomes

\[
\mathbf{G}^{**} = \frac{\varepsilon \mathbf{Z}_1 - \beta \mathbf{Z}_2 - x \mathbf{Z}_3}{\sqrt{x^2 + \beta^2}}, \quad (x \neq 0, \beta \neq 0).
\]  
(2.6)

Separating \( \mathbf{G}^{**} \) into its real and dual parts, we get

\[
\mathbf{G}^{**} = \frac{\varepsilon z_1 - \frac{\beta}{2} z_2 - \frac{x}{10} \mathbf{n}}{\sqrt{x^2 + \beta^2}} + \varepsilon \frac{z_2 - \frac{\beta}{20} z_1 - \frac{x}{20} \mathbf{n}_0}{\sqrt{x^2 + \beta^2}} = \mathbf{g}^{**} + \varepsilon \mathbf{g}^{**}\dagger
\]  
(2.6)'

\[
\mathbf{g}^{**} = \frac{\beta z_1 + x n}{\sqrt{x^2 + \beta^2}} \cdot \mathbf{g}^{**} \dagger + \frac{z_2 - \beta z_1 - x \mathbf{n}_0}{\sqrt{x^2 + \beta^2}}.
\]

From (2.5) and from the above definitions of \( x \) and \( \beta \), we find

\[
\beta = \frac{1}{\lambda} \quad \left( \lambda = \frac{1}{\beta} \right)
\]

(2.7)

\[
x = -\frac{1}{\lambda x} \quad \left( \frac{x}{\beta} = -\frac{q}{\tau} \right).
\]

If we substitute these into \( \mathbf{g}^{**} \), we find

\[
\mathbf{g}^{**} = \mathbf{g}^{**} \dagger.
\]  
(2.8)

Therefore, the following theorem may be stated:
2.3. Theorem

The instantaneous screwing axis $\vec{G}^{**}$ of the trihedron $(\vec{z}_{1}, \vec{z}_{2}, \vec{z}_{3})$, moving along the lines of curvature $\nu = \text{const.}$ on the base surface $\vec{z}$ coincides with $-\vec{X}_{3}$ axis carrying the surface normal $\vec{z}_{3}$ of the trihedron $(\vec{x}_{2}, -\vec{x}_{1}, \vec{z}_{3})$, moving along the lines of curvature $\mu = \text{const.}$ on the base surface $\vec{x}$.

The following theorem may be derived from (2.7):

2.4. Theorem

During the motion considered in Theorem 2.3, the ratio of the geodesic curvatures and the normal curvatures of the surfaces $\vec{x}$ and $\vec{z}$, are equal with a minus sign $\left( \frac{\alpha}{\rho} = - \frac{\beta}{\rho_{n}} \right)$. In other words, the angles between the surface normals and the principal normals of these surfaces are equal with a minus sign $(\text{tg}\omega = - \text{tg}\theta)$.

3. The congruences generated by the instantaneous screwing axes connected with the moving trihedrons of the surfaces $\vec{z}$, $\vec{z}$.

1) Considering $z_{v} = 0$, $\rho_{v} = \tilde{a}$, $\mu_{v} = \tilde{b}$, the instantaneous screwing axis of the moving trihedron $(\vec{z}, -\vec{z}, \vec{n})$, moving along the $\vec{z}$ lines of curvature $\mu = \text{const.}$ on the surface $\vec{z} (u, v)$ is seen in $(2.2)$ and $(2.2)'$. Therefore, the congruence generated by $\vec{G}^{*}$ may be expressed as

$$\vec{y}^{*} = \vec{r} + t^{*} \vec{g}^{*}, \quad (\vec{g}^{*} = 1).$$

(3.1)

Here, the reference surface $\vec{r}$.

$$\vec{r} = \vec{z} + \frac{1}{\tilde{b}} \vec{n}$$

(3.2)
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is the center surface belonging to the line curvature \( u = \text{const.} \) on the surface \( z \) \([2]\).

Therefore, the vectorial equation of the congruence which will be investigated may be stated as

\[
\begin{align*}
\vec{y}^* &= \vec{y}^* (u, v, \vec{t}^*) = z (u, v) + \frac{1}{b (u, v)} \vec{n} (u, v) + \\
&\quad + \vec{t}^* \frac{b (u, v) \vec{z}_1 (u, v) + \vec{a} (u, v) \vec{n} (u, v)}{\sqrt{\vec{a}^2 (u, v) + b^2 (u, v)}}
\end{align*}
\]

2. Considering \( \tau_{gu} = 0 \), \( \rho_{gu} = -z \), \( \rho_{uu} = \beta \), the instantaneous screwing axis of the trihedron \((z, z, \vec{n})\), moving along the lines of curvature \( v = \text{const.} \) on the surface \( z (u, v) \) is seen in \((2.6)\) and \((2.6)'\).

Therefore, the congruence generated by \( \vec{G}^{**} \) may be expressed as

\[
\begin{align*}
\vec{y}^{**} &= \vec{r} + \vec{t}^* \vec{g}^{**}, \quad (g^{**2} = 1).
\end{align*}
\]

Here, the reference surface \( \vec{r} \),

\[
\vec{r} = z + \frac{1}{\beta} \vec{n}
\]

is the central surface belonging to the lines of curvature \( v = \text{const.} \) on the surface \( z \) \([2]\).

The vectorial equation of the line congruence to be investigated may be written as

\[
\begin{align*}
\vec{y}^{**} &= \vec{y}^{**} (u, v, \vec{t}^{**}) = \vec{z} (u, v) + \frac{1}{\beta (u, v)} \vec{n} (u, v) \\
&\quad - \vec{t}^{**} \frac{\beta (u, v) \vec{z}_2 (u, v) + \vec{z} (u, v) \vec{n} (u, v)}{\sqrt{\vec{z}^2 (u, v) + \beta^2 (u, v)}}
\end{align*}
\]

4. Properties of the congruences \( \vec{y}^*, \vec{y}^{**} \).
In order to investigate the properties of the congruences $\vec{y^*}$, $\vec{y^{**}}$ before calculating the first and second fundamental forms, in the KUMMER sense, according to (2.4) and (2.8), (Theorems 2.1 and 2.3), the below theorem may be stated:

### 4.1. Theorem

The congruences $\vec{y^*}$ and $\vec{y^{**}}$ are normal congruences.

1) The first and second fundamental forms of the congruence $\vec{y^*}$ are calculated

\[
\vec{e^*} = \vec{g_u^*} = \vec{g_v^*} = \vec{e_u}^2 = \vec{e_v}^2 = \vec{g_u} = \vec{g_v} = \vec{E} = \vec{G} = r^2 E
\]

\[
\vec{f^*} = \vec{g_u^{**}} = \vec{g_v^{**}} = \vec{f_u}^2 = \vec{f_v}^2 = \vec{f_u} = \vec{f_v} = \vec{G} = \vec{G} = r^2 G
\]

\[
d\vec{\sigma^{**}} = r^2 E du^2 + r^2 G dv^2 = [I^*]
\]  \hspace{1cm} (4.2.)

and

\[
\vec{e^*} = \vec{r_u}^2 \vec{g_u} = \vec{r_u}^2 \vec{g_u} = \vec{r_u} \vec{g_u} = \vec{r_u} \vec{g_u} = \vec{E} = \vec{E} = 0
\]

\[
\vec{f^*} = \vec{r_v} \vec{g_u} = \vec{r_v} \vec{g_u} = \vec{r_v} \vec{g_u} = \vec{r_v} \vec{g_u} = \vec{E} = \vec{E} = 0
\]  \hspace{1cm} (4.3)

\[
\vec{f^{**}} = \vec{r_u} \vec{g_v} = \vec{r_u} \vec{g_v} = \vec{r_u} \vec{g_v} = \vec{r_u} \vec{g_v} = \vec{E} = \vec{E} = 0
\]

\[
\vec{g^*} = \vec{r_v} \vec{g_v} = \vec{r_v} \vec{g_v} = \vec{r_v} \vec{g_v} = \vec{r_v} \vec{g_v} = \vec{E} = \vec{E} = 0
\]

\[
\vec{d} \vec{g^*} = \vec{d} \vec{g^*} = \left( \frac{1}{r} \right) \frac{K}{q} G \vec{d} \vec{G} = \vec{d} \vec{G} = \vec{d} \vec{G} = [I^{**}]
\]  \hspace{1cm} (4.4.)

are found.
2. The first and second fundamental form of the congruence $y^{**}$,

$$
\bar{g}^{**} = \bar{g}_u^{**} = (-\bar{\xi}_u) \cdot \bar{\xi}^2 \bar{E} = \bar{\xi}_1^2 \bar{E} = r^2 \bar{E} \quad (1)
$$

$$
\bar{g}^{**} = \bar{g}_u^{**} \cdot \bar{g}_v^{**} = (-\bar{\xi}_u) \cdot (-\bar{\xi}_v) = \bar{\xi}_1 \cdot \bar{\xi}_2 \sqrt{\bar{\xi}_1 \bar{\xi}_2} \sqrt{\bar{E} \bar{G}} = 0 \quad (2)
$$

$$
\bar{g}^{**} = \bar{g}_v^{**} = (-\bar{\xi}_v) = \frac{\bar{\xi}_2}{2} \bar{G} = \frac{\bar{\xi}_2}{2} \bar{G} = r^2 \bar{G}, \quad (4.5)
$$

$$
d\bar{e}^{**} = r^2 Edu^2 + r^2 Gdv^2 = [I^{**}] \quad (4.6)
$$

and

$$
\bar{e}^{**} = \bar{r}_u \cdot \bar{g}_u^{**} = \bar{r}_u \cdot (-\bar{\xi}_u) = \bar{r}_1 \cdot \bar{\xi}_1 \bar{E} = -\left(\frac{1}{r}\right) \frac{K}{q} \bar{E} \quad (5)
$$

$$
\bar{f}^{**} = \bar{r}_v \cdot \bar{g}_v^{**} = \bar{r}_v \cdot (-\bar{\xi}_v) = \bar{r}_2 \cdot \frac{\bar{\xi}_2}{2} \sqrt{\bar{E} \bar{G}} = \bar{r}_2 \cdot \bar{\xi}_1 \sqrt{\bar{E} \bar{G}} = 0 \quad (6)
$$

$$
\bar{f}^{**'} = \bar{r}_u \cdot \bar{g}_v^{**} = \bar{r}_u \cdot (-\bar{\xi}_v) = \bar{r}_1 \cdot \frac{\bar{\xi}_1}{2} \sqrt{\bar{E} \bar{G}} = \bar{r}_1 \cdot \bar{\xi}_2 \sqrt{\bar{E} \bar{G}} = 0 \quad (7)
$$

$$
\bar{g}^{**} = \bar{r}_v \cdot \bar{g}_v^{**} = \bar{r}_v \cdot (-\bar{\xi}_v) = \bar{r}_2 \cdot \frac{\bar{\xi}_2}{2} \bar{G} = \bar{r}_2 \cdot \bar{\xi}_2 \bar{G} = 0, \quad (8.7)
$$

$$
\bar{d}r \cdot \bar{d}g^{**} = -\bar{d}r \cdot \bar{d}\bar{\xi} = \left(\frac{1}{r}\right)^2 \left(\frac{K}{q}\right) Edu^2 = (\mathbb{I}^{**}) \quad (8.8)
$$

are written.

Also from (4.3) and (4.7), we see that the congruences $y^*$ and $y^{**}$ are normal congruences.

Since (4.2) and (4.6) are same, we may state the following theorem:

4.2. Theorem

The first fundamental forms or the linear elements of their spherical representations of the congruences $\bar{\psi}$ and $\bar{\psi}^{**}$ are equal.
From (4.1) which is identical of (4.5), for $\mathcal{H}^*$ and $\mathcal{H}^{**}$, we find,

$$\mathcal{H}^* = \mathcal{H}^{**} = K^2EG$$  \hspace{1cm} (4.9)

Since $EG$ is always different from zero, we may state the following theorem:

4.3. Theorem

Since the surface $\mathfrak{x}$ cannot be developable, the congruences $\mathfrak{y}^*$ and $\mathfrak{y}^{**}$ cannot be cylindrical congruences.

Since the congruences $\mathfrak{y}^*$, $\mathfrak{y}^{**}$ are normal congruences, for these we may write \(a_2 + a^2 = 0, a \neq 0\) and \(\bar{a}_2 + \bar{a}^2 = 0, \bar{a} \neq 0\) consequitively, which are similar to the conditions \((\bar{q}_1 + q^2 = 0, \bar{q} \neq 0)\) and \((q_2 + q^2 = 0, q \neq 0)\) of the congruences $\mathfrak{y}$ and $\mathfrak{y}$. Since the limit points of the normal congruences $\mathfrak{y}^*$ and $\mathfrak{y}^{**}$ coincide with their focal points,

1) for $\mathfrak{y}^*$, from (4.1) and (4.3) we find

$$\begin{align*}
\bar{l}_{II} &= \bar{\tau}_{II}^* = \frac{1}{1} - \frac{1}{r} = 0 \\
\bar{l}_{II} &= \bar{\tau}_{II}^{**} = 0
\end{align*}$$ \hspace{1cm} (4.10)

2) for $\mathfrak{y}^{**}$, from (4.5) and (4.7), we find

$$\begin{align*}
\bar{l}^{**} &= \bar{\tau}_{II}^{**} = -\frac{1}{r} - \frac{1}{r} = \left(\frac{1}{r} - \frac{1}{r}\right) \\
\bar{l}_{II} &= \bar{\tau}_{II}^{**} = 0
\end{align*}$$ \hspace{1cm} (4.11)

Their middle points become

1) For $\mathfrak{y}^*$,

$$\bar{m}^* = \frac{1}{2} \left(\frac{1}{\bar{r}} - \frac{1}{r}\right),$$  \hspace{1cm} (4.12)
2) For \( \vec{y}^{**} \),

\[
\vec{m}^{**} = \frac{1}{2} \left( \frac{1}{r} - \frac{1}{\bar{r}} \right) = -\vec{m}^*.
\]  \( \text{(4.13)} \)

If we take

\[
\vec{\rho}_r - \vec{\rho}_\Pi = \frac{1}{\bar{r}} - \frac{1}{r} = \vec{\rho}^*
\]

and

\[
\vec{\rho}^{**} - \vec{\rho}_{**} = \frac{1}{\bar{r}} - \frac{1}{r} = \vec{\rho}^{**} = \vec{\rho}^* ,
\]

1) Taking the middle surface

\[
\vec{m}^* = \vec{r} + \vec{\rho}_r \vec{g}^* = \vec{r} + \frac{\vec{\rho}^*}{2} \vec{\zeta}
\]  \( \text{(4.14)} \)

of \( \vec{y}^* \) as the reference surface, the first focal surface generated by the focal point \( \vec{\rho}^* \) becomes

\[
\vec{k}^* = \vec{m}^* + \frac{\vec{\rho}^*}{2} \vec{g}^* = \vec{r} + \frac{1}{\bar{r}} \vec{\zeta} = \vec{r},
\]  \( \text{(4.15)} \)

the second focal surface generated by the focal point \( \vec{\rho}_{**} \) becomes

\[
\vec{p}^* = \vec{m}^* - \frac{\vec{\rho}^*}{2} \vec{g}^* = \vec{r} = \vec{x} + \frac{1}{r} \vec{\zeta} = \vec{z} + \frac{1}{b} \vec{n}.
\]  \( \text{(4.16)} \)

2) Taking the middle surface

\[
\vec{m}^{**} = \vec{r} + \vec{\rho}^{**} \vec{g}^{**} = \vec{r} - \frac{\vec{\rho}^{**}}{2} \vec{\zeta}
\]  \( \text{(4.17)} \)

of \( \vec{y}^{**} \) as the reference surface, the focal surfaces generated by the focal points \( \vec{\rho}^{**}_1 \) and \( \vec{\rho}^{**}_{**} \) become consecutively,

\[
\vec{k}^{**} = \vec{m}^{**} + \frac{\vec{\rho}^{**}}{2} \vec{g}^{**} = \vec{r} + \frac{1}{r} \vec{\zeta} = \vec{r}
\]  \( \text{(4.18)} \)
\[ \vec{p}^{**} = \vec{m}^{**} - \frac{\vec{p}^*}{2}, \quad \vec{g}^{**} = \vec{r} = \vec{x} + \frac{1}{\vec{r}} \vec{z} + \frac{1}{\vec{\beta}} \vec{n}. \quad (4.19) \]

From these the following theorem may be stated:

4.4. Theorem

1°) The distance between the focal points of the congruence \( \vec{y}^* \)
and \( \vec{y}^{**} \) is equal to the distance between the central points of the base
surface \( \vec{x} \), i.e., the focal surfaces of \( \vec{y}^* \) and \( \vec{y}^{**} \) coincide with the
central surface of \( \vec{x} \) (\( \vec{p}^* = \vec{k}^{**} = \vec{r}, \vec{k}^* = \vec{p}^{**} = \vec{r} \)).

2°) One focal surface of \( \vec{y}^* \) and \( \vec{y}^{**} \) each, coincides with the
focal surface of \( \vec{y} \) and \( \vec{y} \) each and become the reference surface
(\( \vec{p} = \vec{p}^* = \vec{k}^{**} = \vec{r}, \vec{p} = \vec{k}^* = \vec{p}^{**} = \vec{r} \)).

On the other hand, from the definition of \( b \) and (2.3) we find

\[ \frac{1}{b} - \frac{1}{\beta} = \rho \quad (4.20) \]

and from the definition of \( \beta \) and (2.7) we find

\[ \frac{1}{\beta} - \frac{1}{\beta} = \rho. \quad (4.21) \]

From these, we find

\[ \vec{k} = \vec{z} + \frac{1}{b} \vec{n} \]

and

\[ \vec{k} = \vec{z} + \frac{1}{\beta} \vec{n}. \quad (4.23) \]

From 4.4. Theorem 2°, (4.16) and (4.18) we get
\[ p = z + \frac{1}{b} \mathbf{n} \]

\[ p = z + \frac{1}{B} \mathbf{n}. \]

Therefore the following theorem may be stated:

4.5. Theorem

The distance between the focal points of the congruences \( \overrightarrow{y} \) and \( \overrightarrow{y'} \) is equal to the distance between the central points of the surfaces \( \overrightarrow{z} \) and \( \overrightarrow{z'} \). i.e. the focal surfaces of \( \overrightarrow{y} \) coincide with the central surfaces of \( \overrightarrow{z} \), the focal surfaces of \( \overrightarrow{y'} \) coincide with the central surfaces of \( \overrightarrow{z} \).

Since the congruences \( \overrightarrow{y^*} \) and \( \overrightarrow{y^{**}} \) are the normal congruences, their principal surfaces which are developable,

1) for \( \overrightarrow{y^*} \)

\[ EG \frac{r_2 K}{q} \, dudv = 0, \quad (4.24) \]

2) for \( \overrightarrow{y^{**}} \)

\[ -EG \frac{r_1 K}{q} \, dudv = 0 \quad (4.25) \]

are found. Since \( EG \neq 0 \), \( r_2 \neq 0 \), \( r_1 \neq 0 \) and \( K \neq 0 \) in (4.24) and (4.25), the following theorem may be stated:

4.6. Theorem

The principal surfaces which are developable of the normal congruences \( \overrightarrow{y^*} \) and \( \overrightarrow{y^{**}} \) are parametric surfaces.

The mean ruled surfaces of the congruences \( \overrightarrow{y^*} \) and \( \overrightarrow{y^{**}} \), we find

1) for \( \overrightarrow{y^*} \)
\[ \overline{\mathcal{C}}^* \, du^2 - \overline{\mathcal{G}}^* \, dv^2 = 0 \]  \hspace{1cm} (4.26)

and

2) for \( y^{**} \)

\[ \overline{\mathcal{G}}^{**} \, dv^2 - \overline{\mathcal{C}}^{**} \, du^2 = 0. \]  \hspace{1cm} (4.27)

Since \( \overline{\mathcal{C}}^* = \overline{\mathcal{C}}^{**} \) and \( \overline{\mathcal{G}}^* = \overline{\mathcal{G}}^{**} \), the following theorem may be stated:

4.7. Theorem

The mean ruled surfaces of the normal congruences \( \overrightarrow{y}^* \) and \( \overrightarrow{y}^{**} \) coincide. As the condition of the congruences \( \overrightarrow{y}^* \) and \( \overrightarrow{y}^{**} \) to become isotropic congruences we find

1) for \( \overrightarrow{y}^* \)

\[- \left( \frac{1}{r} \right)_2 \frac{K}{q} \, G = 0, \]  \hspace{1cm} (4.28)

2) for \( \overrightarrow{y}^{**} \)

\[- \left( \frac{1}{r} \right)_1 \frac{K}{q} \, E = 0. \]  \hspace{1cm} (4.29)

Since \( E \neq 0, \, G \neq 0, \, r_2 \neq 0, \, r_1 \neq 0 \), we may state the following theorem:

4.8. Theorem

Since the base surface \( \overrightarrow{x} \) cannot be developable and canal surface, the congruences \( \overrightarrow{y}^* \) and \( \overrightarrow{y}^{**} \) cannot become isotropic congruences.

REFERENCES
