NOTE ON THE EXISTENCE OF $F$-PERFECT MORSE FUNCTIONS ON COMPACT SURFACES

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ABSTRACT

This paper is dealing with the existence of $F$-perfect Morse functions on a smooth, compact, connected manifold, without boundary. Some concrete results concerning the surfaces $T_g$ and $P_g$, of genus $g \geq 0$, are given.

PRELIMINARIES

Let $M^m$ be a smooth compact, connected manifold of dimension $m \geq 1$, without boundary (i.e. $\partial M = \emptyset$), and let $\mathcal{F}_m(M)$ be the set of all Morse functions defined on $M$. For $f \in \mathcal{F}_m(M)$ let us denote by $\mu_k(f)$ the number of the critical points of $f$ with the Morse index $k$, $0 \leq k \leq m$. Let $\mu(f)$ be the total number of critical points of $f$, i.e.

$$\mu(f) = \sum_{k=0}^{m} \mu_k(f).$$

(1)

The number defined by

$$\gamma(M) = \min \{ \mu(f): f \in \mathcal{F}_m(M) \}$$

(2)

is called the Morse-Smale characteristic of $M$. For more details concerning the above notions we refer to the author's book

[1, Chapter 4].

Because $M^m$ is a compact manifold it follows that $M$ has the homotopy type of a finite CW- complex. Therefore the singular homotopy groups $H_k(M; Z)$, $k = 0, m$, are finitely generated (see Fomenko, A.T. [4, p 94]), that is for $k \in \mathbb{Z}$. 


\[ H_k(M; Z) \simeq (Z \oplus \ldots \oplus Z) \oplus (Z_{n1} \oplus \ldots \oplus Z_{nk}) \]  
\( \beta_k \) times

where \( \beta_k = \beta_k(M; Z) \) are the Betti numbers of \( M \) with respect to the group \((Z, +)\), i.e. \( \beta_k(M; Z) = \text{rank } H_k(M; Z) \), \( k \in Z \).

Consider \( H_k(M; Z) \), \( k = 0, m \), the singular homology groups with the coefficients in the field \( F \) and \( \beta_k(M; F) = \text{rank } H_k(M; F) = \dim_F H_k(M; F) \), \( k = 0, m \), the Betti numbers of \( M \) with respect to \( F \).

Put

\[ \beta(M; Z) = \sum_{k=0}^{m} \beta_k(M; Z), \beta(M; F) = \sum_{k=0}^{m} \beta_k(M; F) \]  

(4)

For \( f \in \mathcal{F}_m(M) \) the following important relations hold:

\[ \mu_k(f) \geq \beta_k(M; F), k = 0, m \]  

(weak Morse inequalities)

\[ \sum_{k=0}^{m} (-1)_k \mu_k(f) = \chi(M) \]  

(Euler formula)

(see Andrica, D. [1, Chapter 3] for the proof and interesting applications). Recall that, in the last relation, \( \chi(M) \) represents the Euler–Poincare characteristic of \( M \), i.e.

\[ \chi(M) = \sum_{k=0}^{m} (-1)^k \dim_{\mathbb{R}} H^k(M), \]  

(5)

where \( H^k(M), k = 0, m \), are the de Rham real cohomology spaces of \( M \).

The Morse function \( f \in \mathcal{F}_m(M) \) is F-perfect if the weak Morse inequalities become equalities, i.e.

\[ \mu_k(f) = \beta_k(M; F), k = 0, m. \]  

(6)

In the sequel we are interested in the following problem, which naturally appears in the theory of the tight and taut immersions (see, for instance, the excellent book of Cecil, T.E., Ryan, P.J. [3]):

**Problem.** For a given field \( F \), characterize the manifolds which admit F-perfect Morse functions.

Concerning this question it is known the following result (see Andrica, D., [1, Chapter 4], [2, Theorem 2]):

**Theorem 1.** The manifold \( M \) has F-perfect Morse functions if and only if
\[ \gamma(M) = \beta(M; F). \] (7)

Let \( p \geq 2 \) be a prime number. Taking into account the relations (3) we can define

\[ d(M, p) = \text{card} \{ n_{k,j}, j = 1, b(k), k = 0, m; p \mid n_{k,j} \} \]

The following result represents a necessary and sufficient condition, in terms of \( \gamma(M), \beta(M; Z) \) and \( d(M, p) \), in order that the manifold \( M \) has \( Z_p \)-perfect Morse functions (see Andrica, D. [2, Theorem 4], [1, Chapter 4]).

**Theorem 2.** The manifold \( M \) has \( Z_p \)-perfect Morse functions an only if the following equality holds

\[ \gamma(M) = \beta(M; Z) + 2d(M, p). \] (8)

The main results. The aim of this note consists in an answer to the above problem when the manifold \( M^m \) is a smooth compact, connected, surface, i.e. the dimension of \( M \) is \( m = 2 \).

Let \( T^2 \) be the 2-dimensional torus, and let us define the smooth, compact, connected, orientable surface of the genus \( g \geq 0 \), by

\[ T_g = T^2 \neq T^2 \neq \ldots \neq T^2, \] (9)

\( g \) times

i.e. \( T_g \) is the connected sum of \( g \) copies of \( T^2 \). If \( g = 0 \), one considers \( T_0 = S^2 \), the 2-dimensional sphere.

Consider \( P_g \) the smooth, compact, connected, and non-orientable surface, of genus \( g \geq 0 \), defined by

\[ P_g = P \cup R^2 \neq P \cup R^2 \neq \ldots \neq P \cup R^2, \] (10)

\( g + 1 \) times

where \( P \cup R^2 \) is the real projective plane.

It is well-known (see Gramain, A. [7]) that, if \( M \) is a smooth, compact, connected surface, without boundary, then \( M \) is diffeomorphic to \( T_g \) if it is orientable, and \( M \) is diffeomorphic to \( P_g \) if it is non-orientable, for some values of \( g \).

The following result is a direct consequence of the well-known exact Mayer–Vietoris sequence in the de Rahm cohomology (see Godbillon, C., [6, Proposition 1.3., p 179]):
\[ \chi (T_g) = 2 - 2g, \quad \chi (P_g) = 1 - g. \] (11)

Kuiper, N.H. [8] (see also [9, § 4] or the book of Cecil, T.E., Ryan, P.J. [3, Proposition 5.6., p 26]) proved the following very interesting connection between the Morse–Smale characteristic given by (2) and the Euler–Poincaré characteristic defined by (5), of a smooth, compact, connected surface \( M \), without boundary:

\[ \gamma (M) = 4 - \chi (M). \] (12)

Using this result and the relations (11) one obtains

\[ \gamma (T_g) = 2 + 2g, \quad \gamma (P_g) = 3 + g \] (13)

**Theorem 3.**

(i) \( T_g \) has \( Q \)-perfect Morse functions.

(ii) For any prime number \( p \geq 2 \), \( T_g \) has \( Z_p \)-perfect Morse functions.

**Proof:** It is well-known (see Lehman, D., Sacré, C. [10, Chapter IV, p 252–302]) that the integer homology of \( T_g \) is given by

\[
H_k (T_g; \mathbb{Z}) \simeq \begin{cases} 
Z & \text{if } k = 0 \\
\bigoplus_{2g} \mathbb{Z} & \text{if } k = 1 \\
Z & \text{if } k = 2 \\
\{0\} & \text{otherwise.}
\end{cases}
\]

One obtains \( \beta_0 (T_g; \mathbb{Z}) = \beta_2 (T_g; \mathbb{Z}) = 1 \), \( \beta_1 (T_g; \mathbb{Z}) = 2g \), and \( d (T_g, p) = 0 \) for any prime number \( p \geq 2 \).

(i) It is known (see Andrica, D. [2, Lemma 3], [1, Chapter 4]) that \( \beta_k (M; \mathbb{Z}) = \beta_k (M; \mathbb{Q}) \), \( k = 0, m; \) thus \( \beta (T_g; \mathbb{Q}) = 2 + 2g \). Taking into account the first relation in (13) it follows \( \gamma (T_g) = \beta (T_g; \mathbb{Q}) \), and the desired conclusion follows from Theorem 1.

(ii) Because \( d (T_g, p) = 0 \), for any prime number \( p \geq 2 \), one obtains \( \gamma (T_g) = \beta (T_g; \mathbb{Z}) + 2d (T_g, p) \), and the conclusion follows via Theorem 2.

**Theorem 4.**

(i) \( P_g \) has not \( Q \)-perfect Morse functions.
(ii) For any prime number $p \geq 3$, $P_g$ has not $Z_p$-perfect Morse functions.

(iii) $P_g$ has $Z_2$-perfect Morse functions.

**Proof:** The singular homology of $P_g$ (see the book of Lehman, D., Sacre, C. [10, Chapter IV, p 252-302]) is

$$H_k(P_g; Z) \simeq \begin{cases} Z & \text{if } k = 0 \\ Z_2 \oplus (Z \oplus \ldots \oplus Z) & \text{if } k = 1 \\ \text{g times} & \\ \{0\} & \text{otherwise.} \end{cases}$$

Therefore $\beta_0(P_g; Z) = 1$, $\beta_1(P_g; Z) = g$, $\beta_2(P_g; Z) = 0$, and

$$d(P_g, p) = \begin{cases} 1 & \text{if } p = 2 \\ 0 & \text{if } p \geq 3. \end{cases}$$

One obtains $\beta(P_g; Z) = 1 + g = \beta(P_g; Q)$. Using the second relation in (13), it follows $\gamma(P_g) = 3 + g = \beta(P_g; Q)$, i.e. $P_g$ has not $Q$-perfect Morse functions. Moreover, $\gamma(P_g) = 3 + g = 1 + g + 2 = \beta(P_g; Z) + 2d(P_g; 2)$, i.e. $P_g$ admits $Z_2$-perfect Morse functions. In an analogous way one can obtain the conclusion (ii).

**Remark.** For $g = 0$ the above results appear in the author's paper [2, Theorems 7, 8] for the $m$-dimensional manifolds $S^m$ and $P \mid R^m$.

**REFERENCES**


