AN APPLICATION OF THE FIBRATION THEOREM OF EHRESMANN

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ABSTRACT

The main purpose of the paper is to prove that the map (7) and also its restriction to $\text{GL}^+ (n, \mathbb{R})$ is a locally trivial fibration.

From the general theory of fiber bundles we know that a bundle map between two $C^\infty$-differentiable manifolds is a surjective submersion. Here arise a natural problem: given $M$ and $N$ two $C^\infty$ - differentiable manifolds and $f: M \rightarrow N$ a smooth surjective submersion, find sufficient conditions in order that $f$ be a locally trivial fibration. A such condition is given by:

Theorem. (Ehresmann [3, Th. 8.12, p. 84]). If $f: M \rightarrow N$ is a proper surjective submersion then $f$ is a locally trivial fibration.

We shall consider

$$\text{GL} (n, \mathbb{R}) = \{ X \in M_n (\mathbb{R}) : \det X \neq 0 \}$$

the real general linear group, which is a $n^2$ dimensional $C^\infty$ – differentiable manifold, as an open subset of $M_n (\mathbb{R})$. It is known that $\text{GL}(n, \mathbb{R})$ has two connected components:

$$\text{GL}^+ (n, \mathbb{R}) = \{ X \in \text{GL} (n, \mathbb{R}) : \det X > 0 \}$$
and

$$\text{GL}^- (n, \mathbb{R}) = \{ X \in \text{GL} (n, \mathbb{R}) : \det X < 0 \},$$

both open in $\text{GL} (n, \mathbb{R})$.

We also consider

$$S_n (\mathbb{R}) = \{ X \in M_n (\mathbb{R}) ; \; ^tX = X \},$$

the set of symmetric matrices. Clearly we can identify $S_n (\mathbb{R})$ with the Euclidean space $\mathbb{R}^{n(n+1)/2}$. In the following we shall denote by $S^+ n (\mathbb{R})$ the subset of $S_n (\mathbb{R})$ formed of all positive definite matrices.
Finally denote by
\[ O_n (IR) = \{ X \in GL (n, IR) : \, ^tX X = I_n \}, \]
the set of orthogonal matrices.

In the sequel we shall use the following two results:

(a) Diagonal form of symmetric matrices. For any \( A \in S_n \) (IR) there exists \( T \in O_n \) (IR) such that
\[ ^tT A T = \begin{bmatrix} \lambda_1 & 0 \\ . & . \\ 0 & \lambda_n \end{bmatrix} \]
where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of the matrix \( A \). (see, for example, [2, Th. 2, p. 83]).

(b) (Polar decomposition in GL (n, IR)). Any \( X \in GL (n, IR) \) admits a unique decomposition in the form:
\[ X = OS \]
with \( 0 \in O_n \) (IR) and \( S \in S_n^+ \) (IR). Moreover the application \( O_n \) (IR) x \( S_n^+ \) (IR) \( \to \) GL (n, IR) given by
\[ (O, S) \longrightarrow O.S \]
is a diffeomorphism.

In this paper, by using the above mentioned result of Ehresmann, we shall obtain a locally trivial fibration of GL (n, IR) (and respectively of \( GL^+ (n, IR) \)) and we will put in evidence an interesting connection with the trivial fibration given by \( \text{det} : GL^+ (n, IR) \longrightarrow IR^* \) (we denote by \( IR^* \) the set of real positive numbers).

Let begin with the proof of two helping results:

**Lemma 1.** The set \( S_n^+ \) (IR) is open in \( S_n \) (IR).

**Proof:** Observe that (a) supply us with the following relation:
\[ S_n (IR) = \left\{ \begin{array}{c} T \{ D (\lambda_1, \ldots, \lambda_n); \, \lambda_i \in IR \} \mid T \in O_n (IR) \end{array} \right\} \]
where
\[ D (\lambda_1, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & . & . \\ . & \ddots & . \\ . & . & \lambda_n \end{bmatrix} \]
In this relation we have
\[ S_n^+(IR) = \left\{ T \{D (\lambda_1, \ldots, \lambda_n); \lambda_i > 0 \text{ for all } i = 1, \ldots, n\} \right\} t_T \]
\[ T \in O_n(IR) \]

But every \( T \{D (\lambda_1, \ldots, \lambda_n); \lambda_i > 0 \text{ for all } i = 1, \ldots, n\} t_T \) is clearly open in \( T \{D (\lambda_1, \ldots, \lambda_n) \lambda_i \in IR\} t_T \); consequently \( S_n^+ (IR) \) is open in \( S_n (IR) \). Lemma 1 is proved.

Consider the following sets: for any \( A \in GL (n, IR) \subset S_n (IR) \) put
\[ O_n (A, IR) = \{ X \in GL (n, IR); \quad ^tXX = A \} \]
and if \( \det A > 0 \)
\[ O_n^+ (A, IR) = \{ X \in O_n (A, IR); \det X = \sqrt{\det A} \} \]
\[ O_n^- (A, IR) = \{ X \in O_n (A, IR); \det X = -\sqrt{\det A} \} \]

Clearly \( O_n (I_n, IR) = O_n (IR) \) and \( O_n^+ (I_n, IR) = SO_n (IR) \) where \( SO_n (IR) \) represents the special orthogonal group.

**Lemma 2.** (i) The map \( \varphi: S_n^+ (IR) \rightarrow S_n^+ (IR) \)
\[ X \varphi (\chi) = \chi^2 \quad (6) \]
is a proper bijection.

(ii) We have the following chain of equivalences:
\[ O_n (A, IR) \neq \emptyset \iff O_n^+ (A, IR) \neq \emptyset \iff A \in S_n^+ (IR). \]

**Proof:** (i) The fact that \( \varphi \) is one-to-one is an immediate consequence of (a). Let's show that \( \varphi \) is proper: for \( K \subset S_n^+ (IR) \) compact we have to prove that \( \varphi^{-1}(K) \) is bounded.

A very useful norm on \( M_n (IR) \), equivalent with the Euclidean norm is
\[ \| A \| = \left[ \max \{ | \lambda_i |; \lambda_i \text{ eigenvalue of } ^tAA \} \right]^{1/2}. \]

But if \( A \in S_n^+ (IR) \) then
\[ \| A \| = \left[ \max \{ \lambda_i^2; \lambda_i \text{ eigenvalue of } A \} \right]^{1/2}. \]
so that \( \| \varphi^{-1}(A) \| = \sqrt{\| A \|}. \) Because \( K \) is bounded, \( \varphi^{-1}(K) \) is bounded, too.

(ii) The first equivalence holds because \( \det (J_n A) = -\det A \), where
\[ J_n = \begin{bmatrix} -1 & 0 \\ 1 & \cdot \\ 0 & 1 \end{bmatrix} \in O_n (IR). \]
Assume $O_n (A, \mathbb{IR}) \neq \emptyset$. Therefore $A = tXX$, thus $A$ is symmetric. If $\lambda$ is an eigenvalue of $A$ and $x \in \mathbb{R}^n$ is an eigenvector corresponding to $\lambda$, then

$$\lambda = \frac{\|X x\|}{\|x\|^2} > 0.$$  

It follows that $A \in S^+_n (\mathbb{IR})$.

Conversely, if $A \in S^+_n (\mathbb{IR})$, by the surjectivity of $\varphi$ one obtains that $O_n (A, \mathbb{IR}) \neq \emptyset$.

Now, we are in position to state the main result of this paper.

**Theorem** (i) The map $f: GL (n, \mathbb{IR}) \to S^+_n$ given by

$$X f (\chi) = t\chi \chi$$

is a fibration of $GL (n, \mathbb{IR})$ with the type fiber $O_n (\mathbb{IR})$.

(ii) The restriction $f \mid_{GL^+ (n, \mathbb{IR})}: GL^+ (n, \mathbb{IR}) \to S^+_n (\mathbb{IR})$

is also a fibration, with the type fiber $SO_n (\mathbb{IR})$.

**Proof:** First, we will show that $f$ is a submersion. Using the well-known result concerning the equality of the Frechet and Gateaux differentials for smooth maps, we obtain:

$$\left( \frac{df}{dC} \right) (C) = \lim_{\lambda \to 0} \frac{1}{\lambda} [f (B + \lambda C) - f (B)] =$$

$$= \lim_{\lambda \to 0} \frac{1}{\lambda} \left[ t(B + \lambda C) (B + \lambda C) - tBB \right] = tBC + tCB.$$  

It follows that, for every $B \in GL (n, \mathbb{IR})$ the differential $(df)_B: M_n (\mathbb{IR}) \to S_n (\mathbb{IR})$ is surjective. Let $D \in S_n (\mathbb{IR})$ and choose $C = t(B^{-1})D / 2$. Then

$$(df)_B (C) = tB \left( t(B^{-1}) D / 2 + tDB^{-1} B / 2 \right) = D / 2 + D / 2 = D.$$  

We prove now that $f$ is a proper map. To this end, let’s observe that, by using the polar decomposition (b) it follows that $f(X) = S^2$. So that $f = \varphi o h$, where $\varphi$ is given by (6) and $h: GL (n, \mathbb{IR}) \to S^+_n (\mathbb{IR})$ is given by
$$h(X) = S.$$  \hfill (8)

By Lemma 2, (i), it remains to show that $h$ is proper. If $K \subseteq S^+_n (\mathbb{I} \mathbb{R})$ is compact then by (5) the set $h^{-1}(K)$ is diffeomorphic to $K \times O_n (\mathbb{I} \mathbb{R})$, so that is compact (don't forget that $O_n (\mathbb{I} \mathbb{R})$ is compact).

Now, we can apply Ehresmann's theorem and deduce that $f$: $GL^+ (n, \mathbb{I} \mathbb{R}) \to S^+_n (\mathbb{I} \mathbb{R})$ is a locally trivial fibration. The fiber along the identity matrix $I_n$ is, as we have already seen, $O_n (\mathbb{I} \mathbb{R})$.

The assertion (ii) follows observing that the restriction of $f$ to the open set $GL^+ (n, \mathbb{I} \mathbb{R})$ is a proper submersion and applying then Ehresmann's theorem.

**Remarks 1.** Notice that both fibrations obtained in Theorem are in fact trivial, since the polar decomposition gives alway a diffeomorphism. In addition, we can give an explicit formula for the fiber along a matrix $A \in S^+_n (\mathbb{I} \mathbb{R})$ for both fibrations:

if $X_0 \in O_n (A, \mathbb{I} \mathbb{R}) \subseteq O_n (A, \mathbb{I} \mathbb{R})$ then the fiber is $O_n (\mathbb{I} \mathbb{R}) X_0 = \{XX_0: X \in O_n (\mathbb{I} \mathbb{R})\}$ for the first fibration, respectively $SO_n (\mathbb{I} \mathbb{R}) X_0$ for the second one.

2. It is worth to mention the following interesting connection between the fibration (ii) and the bundle map

$$g: GL^+ (n, \mathbb{I} \mathbb{R}) \to \mathbb{R}^*_+ , \quad g(X) = (\det X)^2 .$$

The correspondence $X \to (\det X)^2 , \quad \frac{1}{n \sqrt{\det X}} X$ gives a diffeomorphism between $GL^+ (n, \mathbb{I} \mathbb{R})$ and $\mathbb{R}^*_+ \times SL (n, \mathbb{I} \mathbb{R})$ which shows that $g$ is a trivial fibration of the fiber $SL (n, \mathbb{I} \mathbb{R})$. We denote the fiber along $A \in \mathbb{R}^*_+$ by $SL (\mathbb{R}) (n, \mathbb{I} \mathbb{R})$; so that

$$SL (\mathbb{R}) (n, \mathbb{I} \mathbb{R}) = \{X \in GL^+ (n, \mathbb{I} \mathbb{R}) : \det X = \sqrt{\det A}\} .$$

Denoting $S_n (\mathbb{R}) (\mathbb{I} \mathbb{R}) = \{A \in S^+_n (\mathbb{I} \mathbb{R}) : \det A = \alpha\}$ the restriction $f \mid_{SL (\mathbb{R}) (n, \mathbb{I} \mathbb{R})}$ will be a proper submersion. We obtain a fibration of the fiber $SL (\mathbb{R}) (n, \mathbb{I} \mathbb{R})$ with the fiber along $A \in S_n (\mathbb{R}) (\mathbb{I} \mathbb{R})$

$$\{X \in SL (\mathbb{R}) (n, \mathbb{I} \mathbb{R}) : \det X = \sqrt{\det A}\} = \{X \in GL^+ (n, \mathbb{I} \mathbb{R}) : \det X = \sqrt{\det A}\} = SO_n (A, \mathbb{I} \mathbb{R}) ,$$

the same fiber as in (ii).

3. There are serious reasons to believe that some of the results presented here remain valid in the case when $GL (n, \mathbb{I} \mathbb{R})$ is replaced by the automorphism group of an infinite dimensional Hilbert space.