ON THE DEGREE OF APPROXIMATION OF A FUNCTION BY NÖRLUND MEANS OF ITS FOURIER - JACOBI SERIES

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ABSTRACT

In the present paper we prove a theorem on the degree of approximation of a function by Nörlund means of its Fourier-Jacobi series, which generalizes the results of [2] and [3].

1. Let \( \sum a_n \) be any given series with the sequence of partial sums \( \{S_n\}_{n=1}^\infty \). If \( \{P_n\} \) is a sequence of constants, real or complex numbers, such that

\[
P_n = p_0 + p_1 + \ldots + p_n
\]

then the sequence–to–sequence–transformation

\[
t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v S_{n-v}
\]

defines the sequence \( \{t_n\} \) of Nörlund means of the series \( \sum_{n=0}^{\infty} a_n \) generated by the sequence \( \{p_n\} \).

The series \( \sum_{n=0}^{\infty} a_n \) is said be summable by Nörlund means or summable \( (N, p_n) \) to the sum \( S \), if limit \( t_n \) exists and equal to \( S \) as \( n \to \infty \).

2. Let \( F(\theta) = f(Cos \ \theta) \), \( \theta \in [0, \pi] \) be a Lebesque measurable function such that

\[
\int_{0}^{\pi} f(\theta) \ p_n(Cos \ \theta) \ (Sin \ \theta)^{\beta+1} \ Cos \ \theta)^{2\beta+1} \ d\theta
\]
exists, where $\beta > -1$, and $p_n (\cos \theta)$ is the $n$th-Jacobi polynomial of order $(\alpha, \beta)$. The Fourier–Jacobi series associated with this function is given by

$$f (\theta) \sim \sum_{n=1}^{\infty} \hat{f} (n) \, h_n \, R_n (\cos \theta)$$

(2.2)

where

$$\hat{f} (n) = \int_{0}^{\pi} f (\varnothing) \, R_n (\cos \theta) \, d_{\mu} (\varnothing)$$

(2.3)

$$h_n = \int_{0}^{\pi} [R_n (\cos \theta) \, d_{\mu} (\theta)]^{-1}$$

(2.4)

$$R_n (\cos \theta) = \frac{p_n (\cos \theta)}{p_n (1)}$$

(2.5)

and

$$d_{\mu} (\theta) = (\sin \theta)^{2\alpha + 1} (\cos \theta)^{2\beta + 1}$$

(2.6)

Askey and Wainger [1] have defined the convolution structure of two functions $f_1$ and $f_2$ of $L$–class on $[0, \pi]$ in the following manner:

$$(f_1 * f_2) (\theta) = \int_{0}^{\pi} f_1 (\varnothing) \, T_{\varnothing} \, f_2 (\theta) \, d_{\mu} (\varnothing)$$

(2.7)

where the generalisation translation $T_{\varnothing}$ is defined by

$$T_{\varnothing} (\theta) = \int_{0}^{\pi} f (\psi) \, k (\theta, \varnothing, \psi) \, d_{\mu} (\psi)$$

(2.8)

and $K (\theta, \varnothing, \psi)$ is a non-negative symmetric function such that

$$R_n (\cos \theta) \, R_n (\cos \varnothing) = \int_{0}^{\pi} k (\theta, \varnothing, \psi) \, (\cos \psi) \, d_{\mu} (\psi)$$

(2.9)

$$\int_{0}^{\pi} k (\theta, \varnothing, \psi) \, d_{\mu} (\psi) = 1$$

(2.10)
3. Partial sum $S_n(f; \theta)$ of the series (2.2) is given by

$$S_n(f; \theta) = \sum_{v=0}^{n} \hat{f}(v) h_v R_v(\cos \theta)$$

$$= \sum_{v=0}^{n} h_v \int_{0}^{\pi} f(\varnothing) R_v(\cos \theta) R_v(\cos \theta) d\mu \varnothing$$

Now using the orthogonal property of Jacobi polynomials and the relation (2.9) we have

$$S_n(f; \theta) - f(\theta) = \sum_{v=0}^{n} h_v \int_{0}^{\pi} f(\varnothing) k(\theta, \varnothing, \psi) R_n(\cos \psi) d\mu(\varnothing) d\mu(\psi) f(\theta)$$

$$= \sum_{v=0}^{n} h_v \int_{0}^{\pi} \{T \otimes f(\theta) - f(\theta)\} R_n(\cos \psi) d\mu(\psi)$$

$$= B_n \int_{0}^{\pi} \omega_f(\psi) R_n(\cos \psi) d\mu(\psi)$$

(3.1)

where \(\omega_f(\psi) = T \otimes (f(\theta) - f(\theta))\)

(3.2)

and \(B_n = \frac{\gamma(n + \beta + \alpha + 2)}{\gamma(\alpha + 1) \gamma(n + \beta + 1)} \sim n^{\alpha+1}.

Therefore, we have

$$t_n(\theta) - f(\theta) = \frac{1}{P_n} \sum_{k=0}^{n} p_k \{S_{n-k}(f; \theta) - f(\theta)\}$$

$$= \frac{1}{P_k} \sum_{k=0}^{n} p_k B_{n-k} \int_{0}^{\pi} \omega(\psi) p_{n-k}(\alpha+1, \beta)(\cos \psi) d\psi$$

where \(\omega(\psi) = \omega_f(\psi) \left(\sin \frac{\psi}{2}\right)^{2\alpha+1} \left(\cos \frac{\psi}{2}\right)^{2\beta+1}

In 1986 Pandey [3] proved the following theorem

**Theorem A.** Let \(0 < \delta \leq \lambda\), If \(x\) is a point such that

$$\varnothing(t) = \int_{t}^{\delta} \left| \varnothing(u) \right| \frac{1}{u} \left[\frac{1}{n}\right] du$$
\[ = O \left( \begin{bmatrix} \frac{1}{t} \end{bmatrix} g(t) \right) \text{ as } t \to 0 \]

then

\[ t_n(x) - f(x) = O \left( g \left( \frac{1}{n} \right) \right) \]

where \( g(t) \) is a positive increasing function such that

\[ \begin{bmatrix} \frac{1}{t} \end{bmatrix} g(t) \to \infty \text{ as } t \to 0 \]

In (1988) Pathak and Jain [2] proved the following theorem

**Theorem B:** If \( \{p_n\} \) is a non-negative and non-increasing sequence of real or complex numbers, \(-1 \leq \alpha \leq -\frac{1}{2}, \beta > \alpha \) and

\[
\int_t^\infty \frac{\omega(u) p_c \left( \frac{1}{u} \right)}{u^{\alpha + 3/2}} \, du = O(1) \text{ as } t \to 0
\]

then \( L_n(f; \theta) - f(\theta) = O \left( \frac{1}{p_n} \right) \).

The object of the present paper is to generalize the above two theorems A and B in following from.

Our theorem is as follows:

**Theorem:** If \( \{p_n\} \) is a non-negative and non-increasing sequence of real or complex numbers, \(-1 \leq \alpha \leq -\frac{1}{2}, \beta > \alpha \) and

\[
\Phi(t) = \int_t^\infty \frac{w(u) P \left( \frac{1}{u} \right)}{u^{\alpha + 3/2}} \, du = O \left( P \begin{bmatrix} \frac{1}{t} \end{bmatrix} g(t) \right). \quad (4.1)
\]

as \( t \to 0 \)

then

\[ L_n(f; \theta) - f(\theta) = O \left( g \left( \frac{1}{n} \right) \right) \]

where \( g(t) \) is a positive, increasing function such that
\[ P \left[ \frac{1}{t} \right] g(t) \rightarrow \infty \text{ as } t \rightarrow 0. \tag{4.2} \]

We shall use the following lemmas in the proof of our theorem. **Lemma 1:** [2]: Let \( \alpha, \beta \) be real numbers or equal to \(-1\), then

for \( 0 \leq \psi \leq \frac{1}{n} \)

\[ N_n (\psi) = O \left( n^{2\psi+2} \right) \]

where

\[ N_n (\psi) = \frac{1}{P_n} \sum_{k=0}^{n} P_k B_{n-k} P^{(\alpha+1, \beta)}_{n-k} (\cos \psi) \]

**Lemma 2:** [5]: For \( \frac{1}{n} \leq \psi \leq \pi - \frac{1}{n} \),

\[ N_n (\psi) = O \left( \frac{n^{2\psi - \frac{1}{2}}}{P_n} \right) (P \left( \frac{1}{\psi} \right)) (\sin \psi / 2)^{\alpha + \frac{3}{2}} (\cos \psi / 2)^{\beta + \frac{1}{2}} \]

\[ + O \left[ \left( \frac{1}{n} \right)^{\frac{\alpha}{2}} (\sin \psi / 2)^{-\alpha - \frac{5}{2}} (\cos \psi / 2)^{-\beta - \frac{3}{2}} \right] \]

**Lemma 3:** Under the condition (4.1), we have

\[ \int_0^t |\omega(u)| \, du = O \left( t^{\alpha + \frac{3}{2}} g(t) \right) \tag{5.3} \]

**Proof of Lemma 3:** Let

\[ \omega(t) = \int_0^t |\omega(u)| \, P \left[ \frac{1}{u} \right] \, du \]

using the condition (4.1), we have
\[ \int_0^t u \Phi'(u) \, du = \int_0^t |\omega(u)| \, P \left[ \frac{1}{u} \right] \, du \]

on integrating by parts, we get

\[ \omega(t) = O(t) \, P \left[ \frac{1}{t} \right] g(t) + \int_0^t P \left[ \frac{1}{u} \right] g(u) \, du \]

\[ \Rightarrow O(t) \, P \left[ \frac{1}{t} \right] g(t) \]

Thus we have

\[ \int_0^t |\omega(u)| \, du = \int_0^t \frac{w(u)}{P \left[ \frac{1}{u} \right]} \, du \]

\[ \leq \frac{1}{P \left[ \frac{1}{t} \right]} \int_0^t |\omega(u)| \, P \left[ \frac{1}{u} \right] \, du \]

\[ = O \left( \frac{1}{P \left[ \frac{1}{t} \right]} \right) \, O \left( t^{x+\frac{3}{2}} g(t) \right) = O(t^{x+\frac{3}{2}} g(t)) \]

**Proof of the Theorem:** We have

\[ I_n(f; 0) - f(0) = \int_0^\pi \omega(\psi) \, N_n(\psi) \, d(\psi) \]

\[ = \frac{1}{n} \int_0^\pi \omega \left( \frac{1}{n} \psi \right) \, N_n \left( \frac{1}{n} \psi \right) \, d\psi \]

\[ + \int_0^\pi \omega \left( \frac{1}{n} \psi + \frac{1}{n} \right) \, N_n \left( \frac{1}{n} \psi + \frac{1}{n} \right) \, d\psi \]

\[ + \int_0^\pi \omega \left( \frac{1}{n} \psi - \frac{1}{n} \right) \, N_n \left( \frac{1}{n} \psi - \frac{1}{n} \right) \, d\psi \]
we have

\[
I_1 = \int_0^n \omega(\psi) N_n(\psi) \, d\psi
\]

\[= O(\frac{2\pi}{2+2}) O\left(\frac{1}{n^{2+\frac{3}{2}}} g\left(\frac{1}{n}\right)\right)
\]

\[= O\left(g\left(\frac{1}{n}\right)\right) \quad \pi < -\frac{1}{2}
\]

Now, we consider \(I_3\)

\[
I_3 = \int_{\pi-}^{\pi} \frac{1}{n} \omega(\psi) N_n(\psi) \, d\psi
\]

\[
= \int_0^n \omega(\pi-\psi) N_n(\psi) \, d\psi
\]

\[= O(\frac{2\pi}{2+2}) \int_0^n \omega(\pi-\psi) \, d\psi
\]

\[= O(\frac{2\pi}{2+2}) O\left(\frac{1}{n^{2+\frac{3}{2}}} g\left(\frac{1}{n}\right)\right)
\]

\[= O\left(g\left(\frac{1}{n}\right)\right)
\]

Lastly, we consider \(I_2\)

\[
I_2 = \int_{\pi-}^{\pi} \frac{1}{n} \omega(\psi) N_n(\psi) \, d\psi
\]
\[
\begin{align*}
&= O \left( \frac{n^x - \frac{2}{2}}{P_n} \right) \int_{\frac{1}{n}}^{\pi - \frac{1}{n}} \omega(\psi) \left| p \left( \frac{1}{\psi} \right) \right|^x \left( \sin \frac{\psi}{2} \right)^{\frac{3}{2} - \frac{1}{2}} \left( \omega \frac{\psi}{2} \right) \left( \cos \frac{\psi}{2} \right)^{-\frac{3}{2}} d\psi \\
&+ O \left( n^x - \frac{1}{2} \right) \int_{\frac{1}{n}}^{\pi - \frac{1}{n}} \omega(\psi) \left| p \left( \frac{1}{\psi} \right) \right|^x \left( \sin \frac{\psi}{2} \right)^{-\frac{5}{2}} \left( \cos \frac{\psi}{2} \right)^{-\frac{3}{2}} d\psi \\
&= O \left( \frac{n^x + \frac{1}{2}}{P_n} \right) \int_{\frac{1}{n}}^{\pi - \frac{1}{n}} \omega(\psi) \left| p \left( \frac{1}{\psi} \right) \right|^x \left( \sin \frac{\psi}{2} \right)^{\frac{3}{2}} d\psi \\
&+ O \left( n^x - \frac{1}{2} \right) \int_{\frac{1}{n}}^{\pi - \frac{1}{n}} \omega(\psi) \left| p \left( \frac{1}{\psi} \right) \right|^x \left( \sin \frac{\psi}{2} \right)^{\frac{5}{2}} d\psi \\
&= O \left( \frac{1}{P_n} p_n g \left( \frac{1}{n} \right) \right) = O \left( g \left( \frac{1}{n} \right) \right)
\end{align*}
\]

combining the relations I_1, I_2, I_3, we get

\[ I_n (f; \theta) - f (\theta) = O \left( g \left( \frac{1}{n} \right) \right) \]

This completes the proof of the theorem.

**REFERENCES**


