ON THE PARALLEL HYPERSURFACES WITH CONSTANT CURVATURE

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SUMMARY

Gaussian and mean curvatures, \( K_r \) and \( H_r \), for parallel surfaces in \( E^3 \) are given in [2]. In the present note, by means of higher order Gaussian and mean curvatures, we calculate the generalized the curvatures \( K_r \) and \( H_r \) in \( E^{n+1}, n > 2 \).

I. BASIC CONCEPTS

DEFINITION I.1: Let \( M_1 \) and \( M_2 \) are two hypersurfaces in \( E^{n+1} \), with unit normal vector \( N_1 \) of \( M_1 \),

\[
N_i = \sum_{i=1}^{n+1} a_i \frac{\partial}{\partial x_i},
\]

where each \( a_i \) is a \( C^\infty \) function on \( M_1 \). If there is a function \( f \), from \( M_1 \) to \( M_2 \) such that

\[
f: M_1 \rightarrow M_2
\]

\[
P \rightarrow f(P) = (p_1 + ra_1(P), \ldots, p_{n+1} + ra_{n+1}(P)),
\]

then \( M_2 \) is called a parallel hypersurface of \( M_1 \), where \( r \in \mathbb{R} \) [1].

THEOREM I.1: Let \( M_r \) be a parallel surface of the surface \( M \subset E^3 \). Let the Gaussian curvature and mean curvature of \( M \) be denoted by \( K \) and \( H \) at the point \( P \in M \), respectively, and the Gaussian curvature and mean curvature of \( M_r \) be denoted by \( K_r \) and \( H_r \) at the point \( f(P) \in M_r \), respectively. Then we know[1] that

\[
K_r = \frac{K}{1 + rH + r^2K},
\]

and
\[ H_r = \frac{H + 2rK}{1 + rH + r^2K}. \]

**THEOREM 1.2**: Let \( M \) be a hypersurface of \( E^{n+1} \) and \( K_1, K_2, \ldots, K_n \) are the higher order Gaussian curvatures and \( k_1, k_2, \ldots, k_n \) are the principal curvatures at the point \( P \in M \).

Let define of function
\[ \Phi: M \longrightarrow \mathbb{R} \]
\[ P \longrightarrow \Phi(P) = \Phi(r, k_1, k_2, \ldots, k_n) = \prod_{i=1}^{n} (1 + rk_i). \]

Then we have that
\[ \Phi(r, k_1, k_2, \ldots, k_n) = 1 + r \sum_{i=1}^{n} k_i + r^2 \sum_{i<j}^{n} k_ik_j + \ldots + r^n \prod_{i=1}^{n} k_i \]
or
\[ \Phi(r, k_1, k_2, \ldots, k_n) = 1 + rK_1 + r^2K_2 + \ldots + r^n K_n, \]
where \( r \in \mathbb{R} \) is given in definition I.1 [3).

**THEOREM 1.3**: Let \( M_r \) be a parallel hypersurface of the hypersurface \( M \) in \( E^{n+1} \), \( K_1, K_2, \ldots, K_n \) denote the higher order Gaussian curvatures of \( M \), at the point \( P \in M \). \( K_r \) and \( H_r \) are the generalized Gaussian and mean curvatures of \( M_r \), respectively, at the point \( f(P) \in M_r \).

Suppose the function
\[ \Phi: M \longrightarrow \mathbb{R} \]
\[ P \longrightarrow \Phi(P) = \Phi(r, k_1, k_2, \ldots, k_n) = \prod_{i=1}^{n} (1 + rk_i). \]

Then we have
\[ K_r = \frac{\partial^n \Phi(r, k_1, k_2, \ldots, k_n)}{(\partial r)^n (n!) \Phi(r, k_1, k_2, \ldots, k_n)} \]
and
\[ H_r = \frac{\partial \Phi(r, k_1, k_2, \ldots, k_n)}{\partial r \Phi(r, k_1, k_2, \ldots, k_n)} \]
[3].
THEOREM I.4: Let $M$ be a surface of constant positive Gaussian curvature $K$ with no umbilics. Let $r_1 = 1/\sqrt{K}$ and $r_2 = -1/\sqrt{K}$ define parallel sets $M_1$ and $M_2$, respectively. Then $M_1$ and $M_2$ are immersed surfaces of $M$ which have constant mean curvatures $\sqrt{K}$ and $-\sqrt{K}$, respectively. If $M'$ is a surface with constant mean curvature $H$ (non zero) and non zero Gaussian curvature, let $r = -1/H$ yields a parallel set that is an immersed surface of $M'$ with constant positive Gaussian curvature $H^2[2]$.

II. GENERALIZED THEOREMS

THEOREM II.1: Let $M_r$ be a parallel hypersurface of the hypersurface $M$ in $E^{n+1}$. Let $K_1, K_2, \ldots, K_n$ denote the higher order Gaussian curvatures of $M$, at the point $P \in M$ and let

$$\sum_{i=1}^{n-1} r^i K_i = -1$$

then generalized Gaussian curvature of $M_r$ is

$$K_r = \frac{1}{r^n}.$$ 

PROOF: It follows from Theorem I.3 that the generalized Gaussian curvature of a parallel hypersurface is given by

$$K_r = \frac{\partial^n \Phi (r, k_1, k_2, \ldots, k_n)}{(\partial r)^n}$$

$$= \frac{n \prod_{i=1}^{n} k_i}{\prod_{i=1}^{n} (1 + rk_i)}$$

$$= \frac{n \prod_{i=1}^{n} k_i}{1 + rK_1 + r^2K_2 + \ldots + r^{n-1}K_{n-1} + r^nK_n}$$

since we have,

$$\sum_{i=1}^{n-1} r^i K_i = -1$$

then
\[ K_r = \frac{\prod_{i=1}^{n} k_i}{r^n \prod_{i=1}^{n} k_i} \]

or

\[ K_r = \frac{1}{r^n} \cdot \]

Note that there exists a sphere in \( \mathbb{E}^3 \) such that \( \sum_{i=1}^{n-1} r^i K_i = -1 \).

**THEOREM II.2:** Let \( M_r \) be a parallel hypersurface of the hypersurface \( M \) in \( \mathbb{E}^{n+1} \). Let \( K_1, K_2, \ldots, K_n \) denote the higher order Gaussian curvatures of \( M \), at the point \( P \in M \) and let

\[ \sum_{i=1}^{n} (i-1) r^i K_i = 1 \]

then the generalized mean curvature of \( M_r \) is

\[ H_r = \frac{1}{r} \cdot \]

**PROOF:** Theorem I.3 gives us that the generalized mean curvature of a parallel hypersurface \( M \), is given by

\[ H_r = \frac{\partial \Phi (r, k_1, k_2, \ldots, k_n)}{\partial r} \frac{\Phi (r, k_1, k_2, \ldots, k_n)}{\Phi (r, k_1, k_2, \ldots, k_n)} \]

\[ = \frac{K_1 + 2rK_2 + \ldots + nr^{n-1}K_n}{1 + rK_1 + r^2K_2 + \ldots + r^nK_{n-1} + r^nK_n} \]

\[ = \frac{1}{r} \left[ \frac{rK_1 + 2r^2K_2 + \ldots + nr^nK_n}{1 + rK_1 + r^2K_2 + \ldots + r^{n-1}K_{n-1} + r^nK_n} \right] \]

\[ = \frac{1}{r} \left[ \frac{\sum_{i=1}^{n} i r^i K_i}{1 + \sum_{i=1}^{n} r^i K_i} \right] \]

\[ \frac{1}{r} \left[ \frac{1 - \sum_{i=1}^{n} (i-1) r^i K_i}{1 + \sum_{i=1}^{n} r^i K_i} \right] \]

since we have that

\[ \sum_{i=1}^{n} (i-1) r^i K_i = 1 \]

then we get that

\[ H_r = \frac{1}{r} . \]

**COROLLARY:** In the case of \( n = 2 \), Theorem II.1 and Theorem II.2 reduce to the results of [2].

**ÖZET**

**SABİT EĞRİLİKLİ PARALEL HİPERÝÜZEYLER ÜZERİNÉ**

[2] de verilen \( E^3 \) deki paralel yüzeylerin \( K_r \) ve \( H_r \), Gauss ve ortalama eğrilikleri, bu çalışmada, \( n > 2 \) olmak üzere, \( E^{n+1} \) deki yüksek mertebeden Gauss ve ortalama eğrilikleri yardımcıyla genelleştirilmiş ve hesaplanmıştır.

**REFERENCES**

