ON THE BLASCHKE INVARIANTS OF THE AXOIDS OF HELICAL MOTIONS

S. KELEŞ — A.I. SIVRIDAĞ

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ABSTRACT

In this paper the relations between the invariants of the moving axoid $\Sigma$ and the fixed axoid $\Sigma$ under the helical motions of order $k$ in $E^n$ are discussed. Moreover we have the statement (17) for the pair of the 2-ruled surfaces $\tilde{\Gamma} \subset \Sigma$ and $\tilde{\Gamma} \subset \Sigma$ which correspond to each other under the helical motion of order $k$ in $E^n$.

1. HELICAL MOTIONS OF ORDER $k$

A motion of $E^n$ is described in matrix notation by

\[ x = A\tilde{x} + c, \quad AA^T = A^TA = I \]

where $A^T$ is the transposed of the orthogonal matrix $A$ and

\[ A:J \rightarrow O(n), \quad c:J \rightarrow IR^n \]

are functions of differentiability class $C^r (r > 3)$ on a real interval $J$. Considering a motion as a movement of the space $\tilde{E}$ against the space $E$ the co-ordinate vector $\tilde{x}$ in (1) describes a point of so-called moving space $\tilde{E}$ and $x$ a point of the so-called fixed space $E$.

Let $\tilde{x}$ be fixed point in $\tilde{E}$ then (1) defines a parametrized curve in $E$ which is called the trajectory curve of $x$ under the motion. From (1) by differentiating with respect to $t$ we get

\[ \tilde{x} = B(x - c) + \dot{c}, \quad B = AA^T \]

where $B + B^T = 0$, since the matrix $A$ is orthogonal. Therefore in the case of even dimension it is possible that the determinant $|B|$ may not vanish. If $|B(t)| \neq 0$ for all $t \in J$, we get exactly one solution $P(t)$ of the equation

\[ B(t)(P(t) - c(t)) + \dot{c} = 0. \]

The point $P(t)$ is called the pole of the motion at the instant $t$ which is the center of the instantaneous rotation of the motion for $t \in J$. If
\(|B|\) doesn't vanish on \(J\), by considering the regularity condition of the motion we get a differentiable curve \(P: J \mapsto E\) of poles in the fixed space \(E\), called the fixed pole curve. By (1) there is uniquely determined a moving pole curve \(\bar{P}: J \mapsto \bar{E}\) from the fixed pole curve point to point on \(J\).

H.R. Müller proved in [4]; under the motions the fixed pole curve and the moving pole curve are rolling on each other without sliding. Merely in the case \(n = 2\) the motion is determined by the pair of rolling pole curves.

In all other cases (that means \(|B| = 0\), especially for odd \(n\), we obtain by the rules of Linear Algebra that for every \(t \in J\) there exists a unit vector \(e(t) \in \ker B(t)\) and \(\lambda(t) \in \mathbb{R}\) so that the solutions \(y\) of the equation

\[
(4) \quad B(t)(y - c(t)) + \dot{c}(t) = \lambda(t)e(t)
\]

fill a uniquely determined linear subspace \(E_k(t) \subset E^n\) with the dimension \(k = n - \text{rank } B\). \(E_k(t)\) is the axis of the instantaneous screw \((\lambda \neq 0)\) of the motion or the axis of the instantaneous rotation \((\lambda = 0)\) and will be called the instantaneous axis of the motion in \(t \in J\) [1].

If \(|B| = 0\) on the whole interval \(J\) under the regularity conditions we obtain a generalized ruled surface of dimension \(k + 1\) in the fixed space \(E\) generated by the instantaneous axes \(E_k(t)\), \(t \in J\), which we call the fixed axoid \(\mathcal{A}\) of the motion. The fixed axoid \(\mathcal{A}\) determines the moving axoid \(\mathcal{A}'\) in the moving space \(\mathcal{P}\) generator to generator by (1). \(\mathcal{A}'\) and \(\mathcal{A}\) are mapped upon each other by the same values of parameter. In this second case Müller proved in [4]: The axoids \(\mathcal{A}', \mathcal{A}\) of a motion in \(E^n\) touch each other along every common pair \(E_k(t) \subset \mathcal{A}, \mathcal{E}_k(t) \subset \mathcal{A}\) for all \(t \in J\) by rolling and sliding upon each other under the motion. Such a motion is called an (instantaneous) helical motion of order \(k\) in \(E^n\) [1]. A helical motion of order \(k\) is a pure rolling for \(\lambda = 0\).

For the analytical representation of an axoid \(\mathcal{A}\) we choose a leading curve \(y\) in the central (resp. edge) ruled surface \(\Omega \subset \mathcal{A}\) transversal to the generators. In [2] it is shown that there exists a distinguished moving orthonormal frame (ONF) \(\{e_1, e_2, \ldots, e_k\}\) of \(\mathcal{A}\) with the properties:

(i) \(\{e_1, e_2, \ldots, e_k\}\) is an ONF of the \(E_k(t)\),

(ii) \(\{e_{m+1}, e_{m+2}, \ldots, e_k\}\) is an ONF of the central space.

\(z^{k-m}\) (resp. the edge space \(K^{k-m} \subset E_k(t)\))
(iii) $\hat{e}_\sigma = \sum_{\nu=1}^{k} \alpha_{\sigma\nu} e_\nu + \kappa_\sigma a_{k+\sigma}, \quad 1 \leq \sigma \leq m,$

$$\hat{e}_{m+\rho} = \sum_{i=1}^{m} \alpha_{(m+\rho)i} e_i, \quad 1 \leq \rho, \chi \leq k-m,$$

(5) with $\kappa_\sigma > 0$, $\alpha_{\rho\nu} = -\alpha_{\rho\nu}, \alpha_{(m+\rho)(m+\gamma)} = 0$

(iv) $\{ e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m} \}$ is an ONF.

A leading curve $y$ of an axoid $\partial$ is a leading curve of the edge (resp. central) ruled surface $\Omega \subset \partial$ too iff its tangent vector has the form

(6) $\dot{y} = \sum_{\nu=1}^{k} \zeta_\nu e_\nu + \eta_{m+1} a_{k+m+1}$

for $\eta_{m+1} \neq 0$, $a_{k+m+1}$ is a unit vector well defined up to the sign with the property that $\{ e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m}, a_{k+m+1} \}$ is an ONF of the tangent bundle of $\partial$. One shows: $\tau_{m+1}(t) = 0$ in $t \in J$ iff the generator $E_a(t) \subset \partial$ contain the edge space $K^{k-m}(t)$.

Let $\bar{\partial}$ and $\partial$ be the corresponding axoids of the given helical motion of order $k$ in $E^n$ and $\{ \bar{e}_1, \ldots, \bar{e}_k \}$ is a principal ONF of the moving axoid $\bar{\partial}$. Then the equations (iii) hold for $\bar{e}_i$ with barred coefficients, $\bar{\partial}$ has the parameter representation on the interval $J$ by

$\ddot{z}(t, u_1, \ldots, u_k) = \ddot{y}(t) + \sum_{\nu=1}^{k} u_\nu \bar{e}_\nu(t), \quad t \in J, \quad u_\nu \in \mathbb{R}$

where $\ddot{y}$ is a leading curve of the edge (resp. central) ruled surface $\bar{\Omega} \subset \bar{\partial}$.

If we set

(7) $A\bar{e}_\nu = e_\nu, \quad 1 \leq \nu \leq k,$

then we have the following results [1]:

$B e_\nu = 0, \quad 1 \leq \nu \leq k,$

$A \dot{e}_\nu = \dot{e}_\nu$

$A a_{k+\sigma} = a_{k+\sigma}, \quad 1 \leq \sigma \leq m,$

$\alpha_{\rho\nu} = \bar{\alpha}_{\rho\nu}, \quad \kappa_\sigma > 0, \quad 1 \leq \mu, \nu \leq k, \quad 1 \leq \sigma \leq m$

(8) $\eta_{m+1} a_{k+m+1} = \bar{\eta}_{m+1} A a_{k+m+1}, \quad \text{and} \quad |\eta_{m+1}| = |\bar{\eta}_{m+1}|$

$\dot{y} = \Lambda \bar{y} + \lambda e,$

$\zeta_\nu = \bar{\zeta}_\nu + \lambda \bar{y}_\nu, \quad e = \sum_{\nu=1}^{k} \lambda_\nu e_\nu, \quad \| e \| = 1.$
Let a 2-ruled surface (not cylinder) \( \psi \) in \( E^n \) be given by
\[
\psi(t, u) = y(t) + uc(t).
\]
Then the magnitude \( b = \zeta / \kappa \) is called the Blaschke invariant of \( \psi \) where \( \zeta \) and \( \kappa \) are given by (5) and (6) [3].

Let \( \mathcal{D} \) be a \((k+1)\)-ruled surface. The dimension of the asymptotic bundle of \( \mathcal{D} \) being \( k + m \), \( m > 0 \), the magnitudes
\[
b_i = \zeta_i / \kappa_i, \quad 1 \leq i \leq m,
\]
are called the principal Blaschke invariants of \( \mathcal{D} \) and
\[
B = m \sqrt{\prod_{i=1}^{m} b_i}
\]
is called the Blaschke invariant of \( \mathcal{D} \) [5].

In the case \( m = k \) the central ruled surfaces \( \Omega \subset \mathcal{D} \) degenerate in the line of striction. Thus, the Blaschke invariant \( b \) of the 2-ruled surface \( \psi \) generated by the 1-dimensional subspace \( E(t) = \text{Sp} \{e(t)\} \subset E_k(t) \) can be given by
\[
b = \frac{\sum_{\gamma=1}^{k} \zeta_{\gamma} \cos \theta_{\gamma}}{\sqrt{\sum_{\mu=1}^{k - 1} \left[ \left( \sum_{\nu=1}^{k} \cos \theta_{\nu} \zeta_{\nu} \zeta_{\mu} \right)^2 + \left( \cos \theta_{\mu} \kappa_{\mu} \right)^2 \right]}}
\]
where \( e(t) = \sum_{\nu=1}^{k} \cos \theta_{\nu} e_{\nu}(t), \quad \theta_{\nu} = \text{constant}, \quad \|e\| = 1. \) [5]

2. ON THE BLASCHKE INVARIANTS OF THE AXOIDS UNDER THE HELICAL MOTIONS OF ORDER \( k \) IN THE EUCLIDEAN \( n \)-SPACE \( E^n \)

In this section we will discuss the relation between the Blaschke invariants of the moving and fixed axoids (\( m > 0 \)) under the helical motions of order \( k \) in \( E^n \). From (6) we obtain
\[
\zeta_i = \langle \dot{y}, e_i \rangle, \quad 1 \leq i \leq k.
\]
If (8) is considered together with (12) we have
\[
\zeta_i = \langle \dot{\psi}, e_i \rangle + \lambda \lambda_i, \quad e = \sum_{i=1}^{k} \lambda_i e_i.
\]
Thus, From (8), (9) and (13) we get
$$b_1 = \frac{\zeta_1}{k_1} + \lambda \lambda_1 / k_1$$

or

$$b_1 = \bar{b}_1 + \lambda \lambda_1 / k_1.$$  

Hence we have the following results:

**COROLLARY 1.** Let $\varpi$ and $\varphi$ be the moving and fixed axoids (not cylinder) of a helical motion of order $k$ in $E^n$ and $\bar{b}_1$ and $b_1$, $1 \leq i \leq m$, be principal Blaschke invariants of $\varpi$ and $\varphi$, respectively. Then $\bar{b}_1$ the and $b_1$ are generally different and the relation between them is given by (14).

For $\lambda = 0$ which means that the motion is a pure rolling and the Blaschke invariants are $\bar{b}_1$ and $b_1$ agree.

**COROLLARY 2.** The Blaschke invariants $\bar{b}$ of the moving axoid $\varpi$ and $B$ of the fixed axoid $\varphi$ are generally different. If $\lambda = 0$ they agree.

Now that is the point to discuss the relation between the Blaschke invariants of the 2-ruled surfaces $\varphi$ and $\psi$ which correspond to each other generator by generator under the helical motion of order $k$ ($k = m$) such that the ruled surface $\psi$ and the fixed axoid $\varphi$ have the same leading curve $y$ and $\psi$ is generated by the $1$-dimensional subspace $E(t) = Sp \{e(t)\} \subset E_k(t)$. $\bar{b}$ and $b$ being the Blaschke invariants of $\varphi$ and $\psi$, respectively, as in [5].

$$b = \frac{\sum_{\nu=1}^{k} \zeta_{\nu}\cos\theta_{\nu}}{\sqrt{\sum_{\mu=1}^{k} \left[ \sum_{\nu=1}^{k} \cos\theta_{\nu} \zeta_{\nu}\right]^2 + \sum_{\mu=1}^{k} \cos\theta_{\mu} k_{\mu}^2}}$$  

(15)

where $e(t) = \sum_{\nu=1}^{k} \cos\theta_{\nu} e_{\nu}$, $\theta_{\nu} = \text{const.}$ $1 \leq \nu \leq k$.

and $\dot{e}_{\nu} = \sum_{\mu=1}^{k} a_{\nu \mu} e_{\mu}$, $1 \leq \nu \leq k$ [3].

for the helical motions we have

$$<e, e_0> = <\Lambda \bar{e}, \Lambda \bar{e}_0> = <\bar{e}, \bar{e}_0>$$

(16)

$$<\dot{e}_0, e_\mu> = <\Lambda \dot{e}_0, \Lambda \bar{e}_\mu> = <\dot{e}_0, \dot{e}_\mu>.$$

Joining (8), (15) and (16) we get
$$b = \bar{b} + \frac{\sum_{\nu=1}^{k} \lambda_{\nu} \cos \theta_{\nu}}{\sqrt{\sum_{\mu=1}^{k} \left[ \left( \sum_{\nu=1}^{k} \cos \theta_{\nu} z_{\nu \mu} \right)^2 + \left( \cos \theta_{\mu} k_{\mu} \right)^2 \right]}}.$$ 

If we take $e = e_1$, $1 \leq i \leq m$ we obtain (14) from (17). Thus (14) can be considered as a generalization of (17).

**COROLLARY 3.** The Blaschke invariants $\bar{\nu}$ of $\varphi$ and $b$ of $\varphi$ are generally different for the helical motions. For $\lambda = 0$ (pure rolling) $\bar{\nu}$ and $b$ agree. If $\varphi$ and $\varphi$ are 2-dimensional axoids then $\varphi$ and $\varphi$ coincide with $\bar{\varphi}$ and $\varphi$, respectively. In this case, since $\nu = \mu = 1$, $\cos \theta_1 = 1$, $z_{11} = 0$ we obtain $b = \bar{b} + \lambda/\kappa$.

This relation can be obtained from (14) since $\lambda_1 = 1$.

**ÖZET:**

Bu çalışmada $\mathbb{E}^n$, $n$-boyutlu Öklid uzayında k-yıncı mertebeden helisel hareketler altında meydana gelen $\varphi$ ve $\varphi$ hareketli ve sabit aksoidlerinin Blaschke invaryantları arasındaki ilişkiler incelendi. Ayrica bu hareket altında birbirlerine karşılık gelen $\bar{\varphi} \subset \varphi$ ve $\varphi \subset \varphi$ 2-regle yüzey çiftlerinin Blaschke invaryantları arasında bir bağıntı bulundu.

**REFERENCES**


