SOME SPACES OF MATRIX OPERATORS

S.D. PARASHAR

Department of Mathematics University of Delhi, Delhi–110007 INDIA.

(Received Jan. 15, 1991; Revised Aug. 19, 1991; Accepted Nov. 8, 1991)

ABSTRACT

The structural theory of infinite matrices in the classes \((\tau_{p}(p), \tau_{\infty}(p), \ell, f(p), \ell_{m}(p),
\ell_{m}(q), c(p), l_{m}(p))\) or \((c, l_{m}(p))\) have been studied. Some of our results include as a special cases, the
earlier results obtained by Rao.

I. INTRODUCTION

For a sequence \(p = (p_{k})\) of positive real numbers, the following
classes of sequences have been introduced and studied in [1].

\[
\tau(p) = \{ x : \sum_{k=1}^{\infty} |x_{k}|^{p_{k}} < \infty \},
\]

\[
\tau_{\infty}(p) = \{ x : \sup_{k} |x_{k}|^{p_{k}} < \infty \}.
\]

\[
c(p) = \{ x : |x_{k} - t|^{p_{k}} \rightarrow 0 \text{ for some } t \}.
\]

\[
c_{0}(p) = \{ x : |x_{k}|^{p_{k}} \rightarrow 0 \}.
\]

When \(p_{k} = p > 0\), for all \(k\), then \(\tau(p) = \tau_{p}, \tau_{\infty}(p) = \tau_{\infty}, c(p) = c, c_{0}(p) = c_{0}\), where \(\tau_{p}\), \(\tau_{\infty}\), \(c\) and \(c_{0}\) are respectively the spaces of
\(p\)-summable, bounded, convergent and null sequences. In particular, if
\((p_{k}) = \left(\frac{1}{k}\right)\) in \(\tau_{\infty}(p)\) and \(c_{0}(p)\) then these spaces are called spaces of
analytic and entire sequences, respectively. The works on these spaces
has been carried out by Rao in [6], [7], and by other authors. The spaces \(\tau(p), \tau_{\infty}(p), c(p)\) and \(c_{0}(p)\) are linear spaces under coordinatewise addition and scalar multiplication if and only if \(p \in \tau_{\infty}\) see [4].
Let $\lambda$ and $\mu$ be two nonempty subsets of the space $\omega$ of all complex sequences. Then we denote the class of all infinite matrices $A : \lambda \to \mu$ by $(\lambda, \mu)$ such that

$$(A_n(x))_{n=1}^{\infty} = \left(\sum_{k=1}^{\infty} a_{nk}x_k\right)_{n=1}^{\infty} \in \mu,$$

whenever $x \in \lambda$, the convergence of $\sum_{k=1}^{\infty} a_{nk}x_k$ ($n = 1, 2, \ldots$) being assumed.

Recently, the structure theory of infinite matrices transforming spaces of the analytic, entire, bounded and convergent sequences has been studied by Rao [6]. The present paper is devoted to the structural theory of the infinite matrices in the classes $(t_{\alpha}(p), t_{\beta})$, $(t_{\alpha}(p), c)$, $(t(p), t_{\alpha})$, $(c_0(p), t_{\alpha}(q))$ and $(c, t_{\alpha}(p))$. Our results include as a special case, the earlier results obtained by Rao [6]. To find the necessary and sufficient conditions for infinite matrices to be in above mentioned classes one may refer to Chaudhary and Nanda [1].

2. An infinite matrix $A \in (t_{\alpha}(p), t_{\beta})$ if and only if for all integer $N > 1$ we have

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}| \frac{1}{p_k} \leq N < \infty.$$

Let us start with the following theorems:

Theorem 1. Let $p = (p_k) \in t_{\alpha}$ and $N > 1$ be any integer then the class of matrix operators $(t_{\alpha}(p), t_{\beta})$ is a complete metric space with the metric

$$D_N(A, B) = \sup \left\{ \sum_{k=1}^{\infty} |a_{nk} - b_{nk}| \frac{1}{p_k} ; n = 1, 2, \ldots \right\},$$

where $A = (a_{nk}), B = (b_{nk})$ are in $(t_{\alpha}(p), t_{\beta})$.

Proof. It can be proved by the standard arguments that $D_N$ is a metric for every $N > 1$. Finally let $a_k = N^{1/p_k}$ and $A^{(i)}; i = 1, 2, \ldots$ with $A^{(i)} = (a_{nk}^{(i)})$ be a Cauchy sequence in $(t_{\alpha}(p), t_{\beta})$. Then for a given $\varepsilon > 0$, there is a positive integer $i_0$ such that

$$(1) \quad D(A^{(i)}, A^{(j)}) < \varepsilon, \ (i > i_0, j \geq i_0).$$

Since for each fixed $k$ and $n$,
\[ |a_{nk}^{(i)} - a_{nk}^{(j)}| < \sum_{k=1}^{\infty} \alpha_k \left| a_{nk}^{(i)} - a_{nk}^{(j)} \right| < \varepsilon \quad (i \geq i_0, j \geq i_0), \]

therefore \((A^{(i)})\) is a Cauchy sequence of complex numbers and hence converge.

Again \(\frac{\varepsilon}{\alpha_k \cdot 2^k} > 0\), gives the existence of a positive integer \(i_0\), and \(A = (a_{nk})\) such that for each fixed \(k\)

\[ \alpha_k \left| a_{nk}^{(i)} - a_{nk}^{(i)} \right| < \frac{\varepsilon}{2^k}, \quad (i \geq i_0). \]

Thus

\[ \sum_{k=1}^{\infty} \alpha_k \left| a_{nk}^{(i)} - a_{nk}^{(i)} \right| < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} < \varepsilon, \quad (i \geq i_0). \]

It remains to show that \(A = (a_{nk}) \in (t_\infty(p), t_\infty)\).

Letting \(j \to \infty\) in (1), we have

\[ \sum_{k=1}^{\infty} \alpha_k \left| a_{nk}^{(i)} - a_{nk}^{(i)} \right| < \varepsilon, \]

this implies that

\[ \varepsilon \sum_{k=1}^{\infty} \alpha_k \left| a_{nk}^{(i)} - a_{nk}^{(i)} \right| \geq \sum_{k=1}^{\infty} \alpha_k \left| a_{nk}^{(i)} - a_{nk}^{(i)} \right|. \]

Now, \((A^{(i)}) \in (t_\infty(p), t_\infty)\) gives us the required result.

**Corollary 2.** Let \(p = (p_k) \in t_\infty\) and \(Y(p) = (t_\infty(p), c)\), Then the class \(Y(p)\) is a closed subset of \((t_\infty(p), t_\infty)\) and hence a complete metric space with the metric \(D_N\) for each \(N > 1\).

**Proof.** The set \(c\) is a subspace of the BK-space \(t_\infty\), therefore \(Y(p)\) is a subset of \((t_\infty(p), t_\infty)\). Let \(Y(p)\) denotes the closure of \(Y(p)\) in the metric topology \(D_N\). Let \(A \in Y(p)\), then there exists a sequence \((A^{(i)})\) in \(Y(p)\) such that

\[ D_N(A^{(i)}, A) \to 0 \quad \text{as} \quad i \to \infty. \]

Thus for each \(\varepsilon > 0\) there exists \(i_0 > 0\) such that

\[ \sum_{k=1}^{\infty} \alpha_k \left| a_{nk}^{(i)} - a_{nk}^{(i)} \right| < \varepsilon, \quad (i \geq i_0). \]

This implies that
\[ \sum_{k=1}^{\infty} \alpha_k \ |a_{nk}| < \sum_{k=1}^{\infty} \alpha_k \ |a^{(i)}_{nk}| + \varepsilon, \ (i \geq i_0). \]

Hence, \( A = (a_{nk}) \in (\tau_\infty(p), \tau_\infty) \). Finally, to prove \((a_{nk}) \in Y(p)\): \((A^{(i_0)}) \in Y(p)\) gives column limits of \(A^{(i_0)}\) exists, hence for each \(\varepsilon > 0\) there exists a positive integer \(n_0\) such that for each fixed \(k\)

\[ |a_{nk} - a_{mk}| < \frac{\varepsilon}{3\alpha_k 2^k} \ (m \geq n_0, n \geq n_0), \]

then

\[ \sum_{k=1}^{\infty} \alpha_k \ |a_{nk} - a_{mk}| < \frac{\varepsilon}{3}. \]

Now from (1) there is a positive integer \(i_0\) such that

\[ \sum_{k=1}^{\infty} \alpha_k \ |a_{nk} - a_{nk}| < \frac{\varepsilon}{3}. \]

For each fixed \(n\) and \(k\) we have the following,

\[ \sum_{k=1}^{\infty} \alpha_k \ |a_{nk} - a_{mk}| \leq \sum_{k=1}^{\infty} \alpha_k \ |a_{nk} - a_{nk}| \]

\[ + \sum_{k=1}^{\infty} \alpha_k \ |a_{nk} - a_{mk}| + \sum_{k=1}^{\infty} \alpha_k |a_{mk} - a_{mk}| \]

\[ < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \]

Hence

\[ |a_{nk} - a_{mk}| < \frac{\varepsilon}{\alpha_k}, \text{ for all } k. \]

This shows that the column limit of the matrix \(A\) exists. Thus the matrix \(A\) belongs to \(Y(p)\). Arbitrariness of \(A\) in \(Y(p)\) shows that \(Y(p)\) is closed in the complete metric space \((\tau_\infty(p), \tau_\infty)\), which completes the proof.

**Theorem 3.** The space \((\tau_\infty(p), \tau_\infty)\) is separable.

Proof. Let \(M\) denotes the set of all matrices \(B = (b_{nk})\) with rational (complex) entries for which integers \(n_1, q_1\) exists such that \(b_{nk} = 0\) whenever \(n \geq n_1\), or \(k > q_1\) or both, Then \(M\) is a countable subset of \((\tau_\infty(p), \tau_\infty)\). It is sufficient to prove that \(M\) is dense in \((\tau_\infty(p), \tau_\infty)\). Let \(A = (a_{nk})\) be any element of \((\tau_\infty(p), \tau_\infty)\), then for each \(\varepsilon > 0\) there exists \(n_1 > 0\) such that
\[
\sum_{j=n_{j+1}}^{\infty} \frac{1}{p_j} |a_{nj}| N < \frac{\varepsilon}{2}.
\]

Since rationals (complex) are dense in \( \mathbb{C} \), therefore for each entry \( a_{nj} \)
in \( A \) there is a rational entry \( b_{nj} \) close to it. So we can find a matrix
\( B = (b_{nk}) \in M \) satisfying
\[
\sum_{j=1}^{n_{j+1}} |a_{nj} - b_{nj}| N < \frac{\varepsilon}{2}
\]

It follows that
\[
D_N(A,B) = \sum_{j=1}^{n_{j+1}} |a_{nj} - b_{nj}| N \frac{1}{p_j} + \sum_{j=n_{j+1}}^{\infty} |a_{nj} - b_{nj}| N \frac{1}{p_j}
\]
\[
= \sum_{j=1}^{n_{j+1}} |a_{nj} - b_{nj}| N \frac{1}{p_j} + \sum_{j=n_{j+1}}^{\infty} |a_{nj}| N \frac{1}{p_j}
\]
\[
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\]
so \( (\iota_\infty(p), \iota_\infty) \) is separable.

3. For the remainder of this paper \( q = (q_k) \) will denote a sequence of strictly positive real numbers such that
\[
\frac{1}{p_k} + \frac{1}{q_k} = 1 \text{ for all } k.
\]

Let \( Q \) denote the set of all \( p = (p_k) \) for which there exists \( N = N(p) > 1 \) such that
\[
\sum_{k=1}^{\infty} |N^{-1/p_k}| < \infty.
\]

Also it is easy to prove that \( p \in Q \) implies \( p_k \to 0 \) \([2]\).

A more general proof of the following lemma may be found in \([3]\).

Lemma 4. Let \( p \in Q \), then \( A \in (\iota(p), \iota_\infty) \) if and only if
\[
D = \sup_{n,k} |a_{nk}| \frac{q_k}{q_k + 1} < \infty.
\]

Lemma 5\([2]\). Let \( p \in Q \), then \( A \in (c_0(p), \iota_\infty(p')) \) if and only if
\[
\sup_{n,k} \left| a_{nk} \right| < \infty.
\]

Now we prove the following theorems.

Theorem 6. Let \( p \in Q \), then the class of matrix operators \((\mu(p), \mu_{\infty})\) is complete metric space with the metric.

\[
d(A, B) = \sup \{ \left| a_{nk} - b_{nk} \right| \frac{q_k}{q_k + 1}, \ n, k = 1, 2, \ldots \}.\]

Proof. It is obvious that \((\mu(p), \mu_{\infty}), d)\) is a metric space. Now let \( (A^{(i)}) \) be any Cauchy sequence in it, then for each \( \varepsilon > 0 \) there exists a positive integer \( i_0 \) such that

\[
d(A^{(i)}, A^{(j)}) < \varepsilon \quad i, j \geq i_0.
\]

That is,

\[
\left| a_{i,nk}^{(i)} - a_{i,nk}^{(j)} \right| < \varepsilon \quad \frac{q_k}{q_k + 1} \quad i, j \geq i_0.
\]

Hence for each fixed \( n, k \) we have

\[
a_{nk}^{(i)} \to a_{nk} \quad (i \to \infty).
\]

Since \( \varepsilon \cdot \frac{q_k}{q_k} > 0 \), therefore there exists a positive integer \( i_0 \), such that

\[
\left| a_{nk}^{(i)} - a_{nk} \right| < \varepsilon \quad \frac{q_k}{q_k + 1} \quad i \geq i_0.
\]

Thus

\[
d(A^i, A) < \varepsilon \quad (i, j \geq i_0).
\]

Also \( \frac{q_k}{q_k + 1} < 1 \) for all \( k \), and

\[
\left| a_{nk}^{(i)} - a_{nk} \right| \left( \frac{q_k}{q_k + 1} \right) < \varepsilon
\]

gives
\[ \varepsilon > |a_{nk} - a_{nk}| \frac{q_k}{q_k + 1} > |a_{nk}| \frac{q_k}{q_k + 1} - (i) \frac{q_k}{q_k + 1} \]

It follows that

\[ |a_{nk}| \frac{q_k}{q_k + 1} < (i) \frac{q_k}{q_k + 1} + \varepsilon < \infty. \]

Hence, \( A \in (\iota(p), \iota_{\infty}) \).

Theorem 7. Let \( p \in Q \), then the class \((c_0(p), \iota_{\infty}(p'))\) is complete metric space with the metric

\[ d'(A, B) = \sup \{ |a_{nk} - b_{nk}| : n, k = 1, 2, \ldots \} \]

where \( A = (a_{nk}), B = (b_{nk}) \in (c_0(p), \iota_{\infty}(p')) \).

Proof. It can be proved on the lines of Theorem 6. Now if we put \( p_k = \frac{1}{k} \in Q \) for \( N = 2 \) and \( p'_k = \varepsilon \). Then the metric coincide with the metric given by Rao [6].

The following lemma may easily be obtained.

Lemma 8. An infinite matrix \( A \in (c, \iota_{\infty}(p)) \) if and only if \( A \) satisfies

\[ \sup_n \left( \sum_{k=1}^{\infty} |a_{nk}| \right)^{p_n} < \infty. \]

Theorem 9. Let \( \inf p_k > 0 \), then \((c, \iota_{\infty}(p))\) is a complete linear metric space paranormed by \( g_p \) where

\[ g_p(A) = \sup_n \left( \sum_{k=1}^{\infty} |a_{nk}| \right)^{p_n} / M \]

where \( M = \max (1, \sup p_k) \), and \( A = (a_{nk}) \in (c, \iota_{\infty}(p)) \).

Proof. It can be proved by the standard arguments that \( g_p \) is a paranorm and also it is complete. Since, \( g_p(A) = 0 \) implies \( A = 0 \), therefore \((c, \iota_{\infty}(p))\) is a complete linear metric space.

Remark. The condition \( \inf p_k > 0 \) in Theorem 9 cannot be dropped. It follows from the following example:
Example. Let $p_k = \frac{1}{k}$ for all $k$, 

$$A = (A_{nk}) = (\delta_{nk})$$

where $\delta$ is Kronecker. Then $A \in (c, \ell_\infty(p))$. Now consider $0 < |\lambda| < 1$ then $|\lambda|^{-1/k} < 1$ for all $k$ and $|\lambda|^{-1/k} \to 1$ as $k \to \infty$ so that 

$$g_p(\lambda A) = \sup_n \left( \sum_k |\lambda \delta_{nk}| \right)^{1/n}$$

$$= \sup_n (|\lambda|)^{1/n} = 1.$$ 

Hence $\lambda A \not\to 0$ as $\lambda \to 0$ and thus $g_p$ is not a paranorm.

Theorem 10. Let $E \subset (c, \ell_\infty(p))$ be compact then given $\varepsilon > 0$ there is some $i_0 = i_0 (\varepsilon)$ such that for all $n$

$$\left( \sum_{k=i_0}^{\infty} |a_{nk}| \right)^{p_n / M} < \varepsilon$$

for all $A \in E$ and $i \geq i_0$.

Proof. Proof is easy one may see [5].

Acknowledgement. The author is indebted to Prof. P.K. Jain and the referee for his comments.

REFERENCES


[3] Y. LUH; Some matrix transformations between the sequence spaces $\ell(p), \ell_\infty(p), c_0(p), C(p)$ and $w(p)$, Analysis 9 (1989), 67-81.


