A CONVEXITY STUDY IN SPHERE

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ABSTRACT

Convex subsets, convex bodies and foot points in the unit n-sphere $S^n$ are defined. Some geometric properties of convex bodies and foot points in $S^n$ as a manifold with focal points are derived in comparison with the corresponding properties of convex bodies in both Euclidean space $E^n$ and a Riemannian manifold $W$ without focal points.

1- INTRODUCTION

In [8], P.J. Kelly and M.L. Weiss proved some interesting results when they studied the basic geometric properties of convex bodies in the Euclidean space $E^n$. Recently, we studied the same properties when the ambient space is a complete simply connected Riemannian manifold $W$ without focal points [3, 4]. It was found in [3, 4] that almost all the results in $E^n$ concerning convexity, foot points and sectional curvatures are still valid in $W$. As far as we know, properties of convex bodies which have been considered in [3, 4, 8] have not yet been considered for the same sort of bodies in the unit n-sphere $S^n$. Consequently, this work is mainly devoted to study some geometric properties of convex bodies in sphere. Illustrative examples in $E^n$ and $S^n$ are given as a comparison study to show how results are affected by the existence of conjugate points in the ambient space $S^n$.

In the following we give some of the results proved in [3, 4, 8]. Proposition (1-1)

Closed geodesic balls in $W$ are convex bodies. Geodesic spheres in $W$ are closed convex hypersurfaces. Horodiscs in $W$ are convex subsets.
Proposition (1-2)

If \( p \) is an interior point in a convex body \( B \subset W \) (or \( E^n \)) with smooth boundary \( \partial B \), then each geodesic ray from \( p \) intersects the hypersurface \( \partial B \) exactly at one point and the intersection is transversal.

Proposition (1-3)

For a convex body \( B \subset W \) (or \( E^n \)) with smooth boundary \( \partial B \)

(i) each tangent geodesic \( \gamma \) to \( \partial B \) has the property \( \gamma \cap \text{Int} (B) = \emptyset \),

(ii) \( B \) lies on one side of the tangent geodesic hypersurface of \( \partial B \) at each point \( p \in \partial B \)

(iii) no two outer geodesic rays perpendicular to \( \partial B \) meet.

Proposition (1-4)

For a compact smooth hypersurface \( M \) in \( W \) there exists a point \( p \in M \) such that the sectional curvature \( K_p(M) \) of \( M \) at \( p \) satisfies \( K_p(M) \geq K_p(W) \).

The remarkable L. Amaral's theorem [1] can be concluded directly and easily from the last proposition (1-4) if we replace \( W \) by the hyperbolic space \( H^n \) (See [3]).

For other interesting results, specially concerning foot points, we refer the reader to [2, 3, 4, 8]. From now on the unit sphere \( S^n \) is always taken as an imbedded hypersurface

\[
S^n = \{ x : x \in E^n, \langle x, x \rangle = 1 \}
\]

in \( E^n \). Geometric properties of \( S^n \) may be found in any text book in differential geometry. All curves are parametrized by arc length. All manifolds are sufficiently smooth for discussions to make sense.

2- Definitions and Background

Let us begin with introducing the convexity concept in a general Riemannian manifold \( M \).

Definition (2-1)

A set \( A \) in \( M \) is called convex if for each pair of points \( p, q \in A \) there is a unique minimal geodesic segment from \( p \) to \( q \) and this segment is in \( A \).

Definition (2-2)

A set \( B \) in \( M \) is a convex body if it is a compact convex subset of \( M \) with a non-empty interior. A strictly convex body is a convex body
B such that the boundary $\partial B$ of $B$ does not contain any geodesic segment of $M$.

Definition (2-3)

A set $B^o$ is a closed convex (resp. strictly convex) hypersurface of $M$ if it could be made as a boundary of a convex (resp. strictly convex) body $B$ in $M$, i.e $B^o = \partial B$.

In the light of the above definitions we can easily prove the following remarks.

Remarks

(a) The whole of $S^n$ is not a convex set in contrary to the convexity of $E^n$ (or $W$).

(b) Any convex body $B$ in $S^n$ is contained in an open hemisphere $S_{\psi}^n$ of $S^n$ with some point $\psi \in S^n$ as its center.

(c) No two points of a convex body $B \subset S^n$ form a conjugate pair. Same thing is valid in a general Riemannian manifold $M$.

(d) Any convex body $B \subset S^n$ can be mapped geodesically onto a convex body in $E^n$ by using the Beltrami (or central projection) map [2, 6].

(e) Any closed geodesic ball $\overline{B(p, r)}$ centered at an arbitrary point $p \in S^n$ with radius $r < \pi/2$ (small geodesic ball) in $S^n$ is a convex body. This fact is true, in general, for any geodesic ball in either $E^n$ or $W$.

(f) Any closed geodesic ball of radius $r \geq \pi/2$ is not a convex body in $S^n$ and consequently the closed half-space of $S^n$ is not a convex body in contrary to the same property in either $E^n$ or $W$ [3] (By half-space in $W$ we mean horodiscs). A geodesic ball of radius $r = \pi/2$ will be called great geodesic ball.

(g) Any geodesic sphere $S(p, r)$ of center $p$ and radius $r < \pi/2$ (small geodesic sphere) in $S^n$ is a convex hypersurface.

(h) Totally geodesic hypersurfaces (great spheres) in $S^n$ are non-convex hypersurfaces. Also, geodesics of $S^n$ are non-convex subsets of $S^n$.

(i) A geodesic sphere $S(p, r)$ of radius $r$ in $S^n$ has constant sectional curvature $K = 1/\sin^2r$ [2]. In this way, a totally geodesic hyper-sphere in $S^n$ has sectional curvature $K = 1$ while small geodesic spheres in $S^n$ are of constant sectional curvatures greater than 1.
(j) The closure of an open convex subset in $S^n$ is not necessarily convex. Open hemispheres in $S^n$ are good examples of this case. In $E^n$ (or $W$), the closure of any open convex subset is always convex (See [2, 3]). The proof of this fact depends basically on the truncated geodesic cone concept defined below.

(k) Any subset $A$ of $S^n$ with diameter $d(A) \geq \pi$ does not have a convex hull $H(A)$ where $H(A)$ always exists for any subset $A \subset E^n$. If $A \subset S^n$ has $d(A) < \pi$, then $H(A)$ exists. We can show that Beltrami maps from $S^n$ to $E^n$ [2, 6] preserves convex subsets as well as convex hulls.

Definition (2-4)

Let $\gamma$ be a geodesic ray in $S^n$ from $p$. A truncated geodesic cone $pC_\gamma$ in $S^n$ with vertex $p$ and axis $\gamma$ is the family of all geodesic segments emanating from $p$ with the same initial angle with $\gamma$ and each segment is of length less than $\pi$.

The length $\pi$ is excluded in the above definition so as to avoid conjugate points of $p$ on the surface of the cone $pC_\gamma$.

We can give another definition of the cone $pC_\gamma$ as follows.

Consider the exponential map $\exp_p : T_pS^n \to S^n$ restricted to a ball $B(0, r) \subset T_pS^n$ of radius $r < \pi$. This map is a diffeomorphism on $B(0, r)$. The cone $pC_\gamma$ will be taken as the image of a cone $oC_L$ with vertex $0$ and axis the straight line segment $L$ in $T_pS^n$ such that $oC_L \subset B(0, r)$.

One of the main results we shall use later on is the following which relates the height function of a submanifold $N$ in a Riemannian manifold $M$ with its second fundamental form [7].

Proposition (2-1)

Let $N$ be an immersed hypersurface of a Riemannian $n$-manifold $M$. Let $p$ be a point of $N$. Then the second fundamental form of $N$ at $p$ is the Hessian of the height function of $N$ with respect to its tangent space $T_pN$ as a hyperplane of $T_pM$.

3- Main Results on Convexity

Lemma (3-1)

Let $B$ be a convex body in $S^n$ with smooth boundary $\partial B$. If $p$ is an interior point of $B$, i.e $p \in \text{Int}(B)$, then each geodesic ray from $p$ intersects of hypersurface $\partial B$ for the first time transversally.
Proof

Let $\gamma: [0, \infty) \to S^n$ be an arbitrary geodesic ray such that $p = \gamma(0)$. This ray $\gamma$ can not be contained wholly inside $B$ otherwise $B$ will not be contained in an open hemisphere of $S^n$ contradicting remark (b). Assume in contrary to the lemma that $\gamma$ has a tangential first intersection, say $\gamma(a)$, with $\partial B$. Clearly, the geodesic segment $\gamma[0, a]$ is free from conjugate points of $\gamma(a)$. Draw a thin geodesic cone $\gamma(a)C_\gamma$ with vertex $\gamma(a)$, axis $\gamma$ and base $D$ in $B$ (See Fig. (1)). Then there exists a minimal geodesic segment $\overline{\gamma}$ from $\gamma(a)$ to $x \in D$ such that $\overline{\gamma} \not\subset B$ contradicting the convexity of $B$ and the proof is complete.

![Image](image-url)

**Fig. (1)**

Corollary (3–1)

Let $p$ be an interior point of a convex body $B \subset S^n$ with smooth boundary $\partial B$. Let $\gamma: [0, 2\pi] \to S^n$ be a closed geodesic through $p$ such that $p = \gamma(0) = \gamma(2\pi)$. Then the first and the last intersections of $\gamma$ with $\partial B$ are transversal.

Lemma (3–2)

Let $p$ be an interior point of a convex body $B \subset S^n$ and $\gamma: [0, 2\pi] \to S^n$ be a closed geodesic through $p$ such that $p = \gamma(0)$. Let $\gamma(a)$ and $\gamma(b)$, $b > a$, be the first and the last intersections of $\gamma$ with $\partial B$. Then $\gamma(a, b) \cap B = \emptyset$.

Proof

Let $\gamma$ be a closed geodesic through $p$ as given in the lemma. Assume in contrary that $\gamma(a, b) \cap B \neq \emptyset$. Without loss of generality, we consider the case when $\gamma(a, b) \cap B$ is a single point, say $q = \gamma(c)$, for $a < c < b$. (See Fig. (2)).
The geodesic segment $\gamma[a, c]$ should not be minimal otherwise $B$ will be non-convex. Consequently $\gamma[a, c]$ contains a pair $(q_1, q_2)$ of conjugate points. Also, the geodesic segment $\gamma[c, b]$ contains a conjugate pair $(s_1, s_2)$ of points contradicting the geometry of geodesics in $S^n$. The other possibilities can be discussed similarly.

From the above two lemmas we arrive at the following.

Proposition (3–1)

Any closed geodesic $\gamma$ through an interior point $p$ of a convex body $B \subset S^n$ with smooth boundary $\partial B$ intersects $\partial B$ exactly twice. The intersections are all transversal.

Corollary (3–2)

The smooth boundary $\partial B$ of a convex body $B \subset S^n$ is diffeomorphic to $S^{n-1}$.

This corollary can be justified as follows.

Let $B$ be contained in the open hemisphere $S_p^n \subset S^n$ of center $p$. Draw a small geodesic sphere $S(p, \delta)$ of center $p \in \text{Int}(B)$ and sufficiently small radius $\delta$ such that $S(p, \delta) \subset \text{Int}(B)$. By the convexity of $B$ and $B(p, \delta)$, any geodesic ray from $p$ intersects $\partial B$ transversally. In this way, we can build up a central projection diffeomorphism $\beta : \partial B \to S(p, \delta)$. If we compose $\beta$ with $\exp_p^{-1}$ restricted to $S_p^n$, we obtain a diffeomorphism $\exp^{-1} \circ \beta : \partial B \to S(0, \delta) \subset T_p S^n \simeq \mathbb{E}^n$. 
Proposition (3-2)

Let $B \subset S^n$ be a convex body with smooth boundary $\partial B$ and $p \in \partial B$ an arbitrary point. Then $B$ lies on one side of the great hypersphere tangent to $\partial B$ at $p$.

Proof

Assume in contrary to the proposition that $B$ does not lie on one side of the great hypersphere tangent to $\partial B$ at $p$. Consequently, there exists a closed geodesic $\gamma$, with orientation indicated in Fig. (3), tangent to $\partial B$ at $p$ which intersects $\partial B$ at least twice, say at $q$, $r \in \partial B$, transversally (Lemma (3-1)). The geodesic segment $\gamma_{pq}$ -as it lies outside $B$- is not minimal and consequently there exists a point $s \in \gamma_{pq}$ which is conjugate to $p$. The geodesic segment $\gamma_{qp}$ is not contained in $B$ which contradicts the convexity of $B$ and the proof is complete.

![Fig. (3)](image)

Another proof of the above proposition (2-1) may be given if we notice from Fig. (3) that $p$ has two different conjugate points, the first is on $\gamma_{pq}$ and the other is on $\gamma_{rp}$, contradicting the sphere geometry.

The converse of the above proposition is not generally true as any closed hemisphere $S_v^n$ centered at $v \in S^n$ in $S^n$ lies on one side of each tangent great hypersphere to $\partial S_v^n$ while $S_v^n$ is not a convex body in $S^n$. Actually, $S_v^n$ has only one tangent great hypersphere to $\partial S_v^n$ which is itself $\partial S_v^n$. The above proposition (3-2) together with its converse are true in $E^n$ as well as in $W$ [3, 4].

In the light of the above proposition (3-2) and taking into account the property mentioned in proposition (2-1), we have the following consequences.
Corollary (3–3)

For a convex body $B \subset S^n$, each boundary point $p$ is a global minimum point of the height function —with respect to the inner direction and $p$ as a base point— of either $B$ or $\partial B$.

From this corollary, we can easily show that at each point of $\partial B$ the Hessian of the height function as well as the second fundamental form are definite (or semi-definite) [3] and so we have the following results.

Corollary (3–4)

For a convex body $B \subset S^n$, all the boundary points of $B$ have sectional curvature $K$ satisfying $K \geq 1$.

Corollary (3–5)

For a convex body $B \subset S^n$, there exists at least one point $p \in \partial B$ with sectional curvature $K$ strictly greater than 1.

The reason is that $B$ is contained inside a closed ball $\overline{B}(p, r)$ for some $p \in S^n$ and $r < \pi / 2$ where $\partial B \cap \overline{B}(p, r) \neq \emptyset$. For an arbitrary point $q \in \partial B \cap \overline{B}(p, r)$, we have easily —using remark (i) and the height function concept and its relation with sectional curvature—that $K_q$ of $\partial B$ at $q$ satisfies

$$K_q \geq 1 / \sin^2 r > 1.$$ 

Corollary (3–5) can be proved for any closed (not necessarily convex) hypersurface $M$ of $S^n$ such that $M$ is contained in an open hemisphere of $S^n$. In this case, we shall obtain a result similar to that of L. Amaral [1] but in the spherical ambient space.

The following proposition (3–3) gives a necessary and sufficient condition for boundary points of a convex body in $S^n$ to be global maximum points of the height functions.

Proposition (3–3)

For a convex body $B \subset S^n$, each height function on $B$ —with respect to inner direction and boundary base point—has a global maximum point on the boundary $\partial B$ if and only if the diameter $d(B)$ of $B$ satisfies $d(B) \leq \pi / 2$. (The proof is direct)

The following example shows how the condition $d(B) \leq \pi / 2$ is substantial in the above proposition.
Example (3-1)

Let us consider the closed geodesic ball $B(p, r) \subset S^n$ centered at $p \in S^n$ with radius $\pi/4 < r < \pi/2$ as a convex body in $S^n$. Without loss of generality, let us take $B(p, r)$ to be contained in the upper hemisphere of $S^n$ and let $q \in B(p, r) \cap S(v, \pi/2)$ where $v$ is the north pole of $S^n$. It is clear that $v$, which is an interior point of $B(p, r)$ is the global maximum point of the height function based at $q$ (The height of $x \in B$ is $d(x, S(v, \pi/2))$ (See § 4)). Notice that in $E^n$, global maximum points of each height function are always boundary points (See Fig. 4).

![Diagram](image)

Fig. (4-a)  Fig. (4-b)

In Euclidean space $E^n$, it has been proved that for a convex body $B \subset E^n$ all the points of the segment $L$ joining an arbitrary point $p \in \text{Int}(B)$ and a boundary point $q \in \partial B$ are interior points (except $q$.) The corresponding result in $W$ is also valid [4] but in $S^n$ -as a manifold with conjugate (or focal) points - we should add a restrictive condition as follows.

Proposition (3-4).

Let $B$ a convex body in $S^n$, $p \in \text{Int}(B)$ and $q \in \partial B$. Then all the points of the unique minimal geodesic segment $\gamma$ joining $p$ and $q$ are interior points except $q$.

The proof can be carried out by using the truncated geodesic cones concept as it has been done in [3, 4].

The converse of the above proposition (3-4) is not generally true. One might consider any closed hemisphere in $S^n$ as an example. The converse is true if we replace $S^n$ by $E^n$ (or $W$). The reason is that if all
the points of the geodesic segment $\gamma$ joining $p \in \text{Int}(B)$ and $q \in \partial B$ are interior points (except $q$), then $B$ will be a starshaped subset of $E^n$ and $p$ will lie in the kernel $(\text{Ker}(B))$ subset of $B$. In addition, if this is true for all the interior points of $B$, then $\text{Ker}(B) = B$ and consequently $B$ is convex.

4. Foot Points

The distance $d(p, S)$ from a point $p$ to a non-empty subset $S$ of a Riemannian manifold $M$ is defined as [8].

$$d(p, S) = \text{glb} \{d(p, x) : x \in S\}$$

Definition (4-1)

A point $p$ has a foot $q$ in a subset $S \subset M$ if

(i) $q \in S$,

(ii) $q(p, q) = d(p, S)$

One may understand that the foot point of $p$ in $S$ is the nearest point of $S$ to $p$. Moreover, if $\overline{S}$ is the closure of $S$ and $p \notin S$, then $d(p, S) = d(p, \partial S)$, i.e the foot point of a point $p$, in the closure $\overline{S}$ of $S$, is always a boundary point provided that $p \notin S$. If $p \in S$, then $p$ is the unique foot point of itself. Clearly, the foot point of a point $p \notin S$ in a closed subset $S$ with smooth boundary $\partial S$ is a critical point of the distance function $d_p: \partial S \to \mathbb{R}$ defined as $d_p(x) = d(p, x)$. Also the critical geodesic segment from $p$ to the foot point strikes $\partial S$ orthogonally at the foot point (See [5] p. 216).

Proposition (4-1)

Let $B$ be a convex body in $S^n$ and $p \in S^n$ such that $d(p, B) < \pi / 2$. Then $p$ has a unique foot point in $B$.

Proof

If $d(p, B) = 0$, then $p \in B$ and $p$ is the unique foot point of itself as indicated above.

If $0 < d(p, B) < \pi / 2$, then $p \in B$. Assume in contrary that $p$ has two different foot points, say $q_1$ and $q_2$. Let $d(p, q_1) = d(p, q_2) = \iota$, where $\iota < \pi / 2$. Draw the closed geodesic ball $B(p, \iota)$. We have that $B \cap \overline{B(p, \iota)} = \{q_1, q_2\}$. As $B(p, \iota)$ has diameter less than $\pi$, then $\overline{B(p, \iota)}$ is a convex body in $S^n$. In this way, the unique minimal geodesic
segment $\gamma$ joining $q_1$ and $q_2$ is contained in $\overline{B(p, r)}$. Hence, $\gamma \cap B$ which contradicts the convexity of $B$. A similar discussion can be carried out if $p$ has more than two foot points.

Remark (4–1)

If $d(p, B) \geq \pi/2$, then the above proposition (4–1) is not necessarily true in the light of the following example.

Example (4–1)

Consider $B$ to be a convex body in $S^2$ which has a part $\gamma_1$ of its smooth boundary $\partial B = \gamma_1 \cup \gamma_2$ as a sufficiently small geodesic segment. The south pole $p$ of $S^2$ which satisfies $d(p, B) = \pi/2$ has all the points of $\gamma_1$ as foot points. (See Fig. (5)).

We can show that the above proposition (4–1) is still true for $d(p, B) \leq \pi/2$ on the condition that $B$ is strictly convex.

The concept of the foot points in subsets as well as convex subsets in sphere (or more generally in a Riemannian manifold) deserves to devote a separate work. We hope to achieve such a study in the near future.

REFERENCES


