THE SETS OF HOMOTHETIC MAPPINGS

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ABSTRACT

In this work, the homothetic Matrix Lie Group has been considered as an action group and the homothetic mapping sets have been obtained as a subset of mapping sets on \( \mathbb{R}^n \).

1. INTRODUCTION

Consider that \( G \) is a group and \( M \) is a differentiable manifold. As a consequence,

(a) The points on \( M \) coincide with elements of \( G 

(b) \quad o : \quad M \times M \longrightarrow M 

\quad (a, b) \quad \longrightarrow \quad ab^{-1} 

this operation is also differentiable in every where. \((M, G)\) representation which has these two axioms is called a Lie Group [1].

If

\[ \{[a_{ij}]_{n \times n} \mid a_{ij} \in \mathbb{R} \} \]

is a submanifold of matrix space and a group with respect to matrix multiplication, then this group is defined as a matrix Lie Group [2].

Let \( M, \bar{M} \) be \( n \)-dimensional \( C^\infty \) — manifolds and

\[ \varphi : M \xrightarrow{\text{diffeomorphism}} \bar{M}, \]

such that

\[ \varphi_* : TM \longrightarrow T\bar{M}, \quad \forall \ x, \ y \in T_M(p) \]

and

\[ \varphi_*(x, \varphi_*(y)) = c^2 \quad \langle x, y \rangle |_p \]

where \( c^2 \) is a constant.

The transformation \( \varphi \) which satisfies above equality is defined as a Homothetic Transformation [3].
Since homothetic transformations are free of metric choice, there is no need to any specialization in the metric.

If $A$ is an orthogonal $n \times n$ matrix and $k = cI_n$ is a scalar matrix, then

$$H = kA,$$

is called a homothetic matrix.

The set of homothetic transformations $(H(M))$ is a group with respect to the operation of composition of functions. The set of homothetic matrices $(\mathcal{H}(M))$ which corresponds to the set of homothetic transformations $(H(M))$ is also a group with respect to matrix multiplication. Thus, the set $(H(M))$ which corresponds to the set $(\mathcal{H}(M))$ is a group isomorphism [4].

The set of homothetic matrices $(\mathcal{H}(M))$ is also a Matrix Lie Group [4].

2. MAPPING ON $\mathcal{H}(E^n)$

Definition (Homothetic mapping): Let $E^n$ be an n-dimenstional $C^\infty$ — manifold and $(U, \varphi)$ be a coordinate neighborhood. Then, there exist such functions;

$$f_x = \{h_1|_x, h_2|_x, \ldots, h_n|_x ; x\}, \forall x \in \varphi(U), f_x \in \mathcal{H}B(E^n),$$

$$h_i|_x = \sum_{k=1}^{n} c a_{ki} \frac{\partial}{\partial x_k} |_x .$$

The linear mapping $(f_x)$ is called a homothetic mapping on $E^n$.

Theorem 1: $(B(E^n) (E^n, GL(n,IR)))$ is given as a main fibre set. Then, the following transformation exists:

$$V \subset E^n, \varphi : \pi^{-1}(V) \longrightarrow V \times GL(n,IR).$$

By means of above transformation, homothetic mapping converges to a homothetic matrix. In other words, every homothetic matrix indicates a homothetic mapping.

Proof: Let

$$f_x \in \mathcal{H}B(E^n) \ni \{h_1|_x, h_2|_x, \ldots, h_n|_x ; x\}$$

then, one obtains that
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\[ f_x \rightarrow \psi(f_x) = (x, [x_{ki}]), \quad h | x = \sum_{k=1}^{n} x_{ki} \quad \frac{\partial}{\partial x_k} | x. \]

In fact,

\[ \mathcal{H} B(E^n) \subset B(E^n). \]

Thus, one can say that \([x_{ki}] \in \text{GL} (n, IR).\]

\[ f_x : IR^n_1 \xrightarrow{\text{Lineer}} T_{E^n}(x) \]

\[ f_x P = [x_{ki}] p \]

\[ f_x P = \begin{bmatrix}
\sum_{i=1}^{n} x_{1i}p_i \\
\vdots \\
\sum_{i=1}^{n} x_{ni}p_i
\end{bmatrix} = \left( \sum_{i=1}^{n} x_{1i}p_i \right) \left. \frac{\partial}{\partial x_i} \right|_x + \cdots + \left( \sum_{i=1}^{n} x_{ni}p_i \right) \left. \frac{\partial}{\partial x_n} \right|_x. \]

Using the above equality, we can write

\([x_{ki}] = [ca_{ki}], \ l \leq i, k \leq n \]

or, in other way,

\([ca_{ki}] \in \mathcal{H} (E^n) \]

if \([ca_{ki}] \in \mathcal{H} (E^n)\) is given

then

\([ca_{ki}] \in \text{GL} (n, IR). \]

Thus,

\[ \exists f'_x \in B(E^n) \exists f'_x = \{h'_1 | x, \ldots, h'_n | x; x\} \]

where

\[ h_l | x = \sum_{k=1}^{n} ca_{ki} \left. \frac{\partial}{\partial x_k} \right|_x. \]

Finally, we can write

\[ f'_x \in \mathcal{H} B(E^n). \]
Theorem 2: Let $x$ be an any point on the n-dimensional Euclidean space $E^n$. If $\varphi$ is a homothetic transformation of $E^n$ then there is a radial transformation $r$ of $E^n$ and a rotation $g$ around $x$ and a sliding $t$ (or another sliding $t'$) of $E^n$, such that

$$\varphi = \text{torog or } \varphi = \text{rogot'}.$$ 

Proof: Let an orthogonal system with initial point $x$ at $E^n$ be

$${x_1, x_1, \ldots, x_n}$$

and a homothetic transformation be $\varphi$. Using the orthogonal system, homothetic motion, with matrix representation, will be,

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} kA & B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad k = cI_n \in O(n), \ A \in O(n), \ B \in \mathbb{R}^n_1$$

and using the fact that $D = \frac{1}{c} A^{-1} B$ one can obtain

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} cI_n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{c} kD^T \\ 0 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}.$$ 

In the above equality, the first left matrix represents a scalar matrix $k = cI_n$, which gives us a radial transformation $r$. Second matrix defines a rotation around the point $x$ and the third matrix indicates a sliding of $E^n$ which is defined by

$$D = \frac{1}{c} A^{-1} B.$$ 

So we can write that $\varphi = \text{rogot'}$.

One can shows that the set of homothetic motions $\mathcal{H}(n)$ is a group with respect to the matrix multiplication.

Theorem 3:

For $x, y \in E^n$ and $f_x, f_y \in \mathcal{H} B(E^n)$ there is only one homothetic motion $\varphi$ such as

$$\varphi (f_x) = f'_y.$$ 

Proof: Let

$$f_x = \{h_1 | x, h_2 | x, \ldots, h_n | x; x \} \ ; f'_y = \{h'_1 | y, \ldots, h'_n | y; y \} \in \mathcal{H}(B(E^n))$$
where $\varphi$ denotes the homothetic motion,

$r$ denote: the radial transformation,

$g$ denotes the orthogonal transformation,

$t$ denotes the sliding motion.

By using the theorem 2, one can write

$$\varphi = \text{torog.}$$

On the other hand, by using the technique given in [1], one obtains (in the following figure)

$$t(x) = y, \text{ when } t \in T(n),$$

similarly, for only one rog,

$$t_*^{-1}(h'_1) = (\text{rog})_*(h_1)$$

or

$$h'_1 = t_*(\text{rog})_*(h_1),$$

$$h'_i = (\text{torog})_*(h_i),$$

$$h'_i = \varphi_*(h_i).$$

Thus we can write that

$$\varphi(\{h_1 | x, \ldots, h_n | x; \ x\}) = \{\varphi_*(h_1 | \varphi(x)), \ldots, \varphi_*(h_n | \varphi(x)); \ \varphi(x)\} = \{h'_1 | y, \ldots, h'_n | y; y\}$$

$$\varphi(f_x) = f'_y.$$

This result shows us the availability of a homothetic motion $\varphi$ and its singularity.
REFERENCES


