GENERALIZED CROSS PRODUCT in $\mathbb{R}^6$ and $\mathbb{R}^m$, $m = \frac{n(n-1)}{2}$

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ABSTRACT

In this study in the space $\mathbb{R}^6$, the cross–product was defined as analogous vector–product in $\mathbb{R}^3$. We showed that this product makes $\mathbb{R}^3$ a Lie algebra. Therefore, it was showed that the Lie algebras $(\mathbb{R}^3, 0)$ and $(A_3, [\cdot, \cdot])$ are isomorphic. As a generalization, in the space of dimension $m = n (n-1)/2$, cross–product can be given as

$$R^m \times R^m \rightarrow R^m, \ x o y = J^{-1} [J(X), J(Y)]$$

where $J = R^m \rightarrow An$ is Lie algebra isomorphism. At the end, we showed that the cross–product we defined is vector product well known for $n = 3$.

INTRODUCTION

Studying kinematics, the set $A_3 = \{A \in M (3x3, R)/A^{-1} = A^T\}$ is very important. $A_3$ is the Lie algebra with the product $[A, B] = AB–BA$. And $(A_3, [\cdot, \cdot])$ is the Lie algebra of orthogonal matrices of order $3 \times 3$, $0(3)$. Moreover $(R^3, x)$ is the other Lie algebra and this algebra is isomorphic to $(A_3, [\cdot, \cdot])$ [1]. This isomorphism gave to use an inspiration. Is it possible to define a product in $A_m$, $m = \frac{n (n-1)}{2}$, which makes $R^m$ a Lie algebra under a isomorphism? We showed that it is possible. At the first we studied in $R^6$ (which is isomorphic to $A_4$).

The product we will define is handy to study the theory of dual numbers and of the dual sphere [2]. For this purpose we used the properties of anti-symmetric mappings, permutations and determinant function [3].

2. Let $S_6$ be permutation group of the set $M = \{1, 2, \ldots, 6\}$. We define a relation on $S_6$ as following. For every $\sigma, \lambda$ are elements of $S_6$,
\( \sigma \prec \lambda \rightarrow \sigma(5) = \lambda(5) \) and \( \sigma(6) = \lambda(6) \)

This relation is a equivalence relation on \( S_6 \). So we have equivalence class. Each equivalence class is known by an element of the set

\[
PC_2^6 = \{(1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)\}
\]

For \((\lambda, \beta) \in PC_2^6\), the equivalence class is shown by \(\sigma_{\lambda, \beta}\). Half of the permutations of \(\sigma_{\lambda, \beta}\) are even the others are odd. So \(\sigma_{\lambda, \beta} = (\sigma_{\lambda, \beta})_e \cup (\sigma_{\lambda, \beta})_o\). But we will use only even permutations and write \(\sigma_{\lambda, \beta}\) for \((\sigma_{\lambda, \beta})_e\).

3. Cross product in \( R^6 \)

Let \(\{e_1, e_2, e_3, e_4, e_5, e_6\}\) be the standard base of \( R^6 \). We define the product \( \otimes \) as the following.

<table>
<thead>
<tr>
<th>(\otimes)</th>
<th>(e_1)</th>
<th>(e_2)</th>
<th>(-e_5)</th>
<th>(e_2)</th>
<th>(e_3)</th>
<th>(0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e_1)</td>
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<td>(e_4)</td>
<td>(-e_5)</td>
<td>(e_2)</td>
<td>(e_3)</td>
<td>(0)</td>
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<tr>
<td>(e_2)</td>
<td>(e_4)</td>
<td>0</td>
<td>(-e_6)</td>
<td>(-e_1)</td>
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<td>(e_3)</td>
</tr>
<tr>
<td>(e_3)</td>
<td>(e_5)</td>
<td>(e_6)</td>
<td>0</td>
<td>0</td>
<td>(-e_1)</td>
<td>(-e_2)</td>
</tr>
<tr>
<td>(e_4)</td>
<td>(-e_2)</td>
<td>(e_1)</td>
<td>0</td>
<td>0</td>
<td>(-e_6)</td>
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<td>(e_5)</td>
<td>(-e_3)</td>
<td>0</td>
<td>(e_1)</td>
<td>(e_6)</td>
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<td>(-e_4)</td>
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<tr>
<td>(e_6)</td>
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<td>(-e_3)</td>
<td>(e_2)</td>
<td>(-e_5)</td>
<td>(e_4)</td>
<td>0</td>
</tr>
</tbody>
</table>

Tableau 1.

If \(\sigma \in \sigma_{(\lambda, \beta)}\), then \(e_{\sigma(5)} \otimes e_{\sigma(6)} = e_\lambda \otimes e_\beta\). Also we have

\[
\Sigma_{\sigma \in \sigma_{\lambda, \beta}} \det [X, Y, e_{\sigma(1)}, \ldots, e_{\sigma(4)}] e_{\sigma(5)} \otimes e_{\sigma(6)} = 12 \det [X, Y, e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}, e_{\sigma(4)}] e_\lambda \otimes e_\beta
\]

where \(\sigma\) is an element of \(\sigma_{(\lambda, \beta)}\). In such a manner that, \(\det [X, Y, e_{\sigma(1)}, \ldots, e_{\sigma(4)}] = \)

\[
\begin{bmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\
\sigma(1) & \delta & \sigma(2) & \delta & \sigma(3) & \delta \\
\sigma(2) & \delta & \sigma(3) & \delta & \sigma(4) & \delta \\
\sigma(3) & \delta & \sigma(4) & \delta & \sigma(5) & \delta \\
\sigma(4) & \delta & \sigma(5) & \delta & \sigma(6) & \delta \\
\sigma(5) & \delta & \sigma(6) & \delta & \sigma(1) & \delta \\
\sigma(6) & \delta & \sigma(1) & \delta & \sigma(2) & \delta \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
  x_{\sigma(1)} & x_{\sigma(2)} & x_{\sigma(3)} & x_{\sigma(4)} & x_{\sigma(5)} & x_{\sigma(6)} \\
  y_{\sigma(1)} & y_{\sigma(2)} & y_{\sigma(3)} & y_{\sigma(4)} & y_{\sigma(5)} & y_{\sigma(6)} \\
  1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

\[
= x_{\sigma(5)} y_{\sigma(6)} - x_{\sigma(6)} y_{\sigma(5)}. 
\]

So we have

1. Definition: For every \( X, Y \in \mathbb{R}^6 \) the product

\[
X \times Y = \sum_{\sigma} \det [X, Y, e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}, e_{\sigma(4)}] e_{\sigma(5)} \otimes e_{\sigma(6)}
\]

is called cross-product in \( \mathbb{R}^6 \).

2. Proposition: If \( X, Y \in \mathbb{R}^6 \) then

1) \( X \otimes Y = -Y \otimes X, X \otimes X = 0 \) (anti-symmetry-property)

2) The product \( \otimes \) is bilinear.

Proof: It is easy to show by using the properties of determinant function.

3. Proposition: If \( X, Y \in \mathbb{R}^6, \sigma \in \sigma_{2,3} \) and \( (\lambda, \beta) \in P_{2,3} \) then

\[
X \otimes Y = \sum_{\sigma} (X_{\sigma(5)} Y_{\sigma(6)} - X_{\sigma(6)} Y_{\sigma(5)}) e_{\sigma(5)} \otimes e_{\sigma(6)}
\]

With the direct calculation, \( X \otimes Y \) can be written in component of \( X \) and \( Y \) as follows

\[
X \otimes Y = (x_4 y_2 - x_4 x_2 + x_5 y_3 - y_3 x_2) e_1 + (x_1 y_4 - y_1 x_4 + x_6 y_3 - y_3 x_3) e_2 \\
+ (x_1 y_5 - y_1 x_5 + x_2 y_6 - y_2 x_6) e_3 + (x_2 y_1 - y_2 x_1 + x_6 y_5 - y_5 x_5) e_4 \\
+ (x_4 y_6 - x_4 x_6 + x_3 y_1 - y_3 x_1) e_5 + (x_3 y_2 - y_3 x_2 + x_5 y_4 - y_5 x_4) e_6
\]

4. Theorem: The product \( \otimes \) in \( \mathbb{R}^6 \) has the following properties.

1) For every \( X, Y \in \mathbb{R}^6 \),

\[
< X \otimes Y, X > = 0, < X \otimes Y, Y > = 0
\]
2) For every $X, Y, Z \in \mathbb{R}^6$
\[<X \otimes Y, Z> = <X, Y \otimes Z>\]

3) For every $X, Y, Z \in \mathbb{R}^6$
\[X \otimes (Y \otimes Z) + Y \otimes (Z \otimes X) + Z \otimes (X \otimes Y) = 0\]

Proposition 2 and theorem 3 show that $(\mathbb{R}^6, \otimes)$ is a Lie algebra.

5. Isomorphism between the Lie algebras $(\mathbb{R}^6, \otimes)$ and $(A_4, [,])$
Let $A_4$ be the set of all anti-symmetric matrices of order 4x4. The system $(A_4, \oplus, (R, +, .), \otimes)$ is a vector space of dimensions 6 and $A_4$ is a Lie algebra with the Lie bracket operator $[,].$

\[[,] = A_4 \times A_4 \rightarrow A_4, [X, Y] = X.Y - Y.X\]

Therefore we have the Lie algebra $(\mathbb{R}^6, \otimes)$. We can define a mapping $J$ between $\mathbb{R}^6$ and $A_4$ as follows.

\[J = \mathbb{R}^6 \rightarrow A_4\]

\[J(x_1, x_2, x_3, x_4, x_5, x_6) = \begin{bmatrix}
0 & x_1 & x_2 & x_3 \\
-x_1 & 0 & x_4 & x_5 \\
-x_2 & -x_4 & 0 & x_6 \\
-x_3 & -x_5 & -x_6 & 0
\end{bmatrix}\]

5. Theorem: The mapping $J$ is a Lie algebra isomorphism.

Proof: The mapping $J$ is one-to-one and onto. Moreover, for

\[X = \sum_{i=1}^{6} x_i \cdot e_i, \quad Y = \sum_{i=1}^{6} y_i \cdot e_i,\]

\[J(X \otimes Y) = J(x_4y_2 - y_4x_2 + x_5y_3 - y_5x_3, x_1y_4 - y_1x_4 + x_6y_3 - y_6x_3, x_1y_5 - y_1x_5 + x_2y_6 - y_2x_6, x_2y_1 - y_2x_1 + x_6y_5 - y_6x_5, x_4y_6 - y_4x_6 + x_3y_1 - y_3x_1, x_3y_2 - y_3x_2 + x_5y_4 - y_5x_4)\]

\[= \begin{bmatrix}
0 & x_1 & x_2 & x_3 \\
-x_1 & 0 & x_4 & x_5 \\
-x_2 & -x_4 & 0 & x_6 \\
-x_3 & -x_5 & -x_6 & 0
\end{bmatrix} = \begin{bmatrix}
0 & y_1 & y_2 & y_3 \\
y_1 & 0 & y_4 & y_5 \\
y_2 & -y_4 & 0 & y_6 \\
y_3 & -y_5 & -y_6 & 0
\end{bmatrix} = XY - YX\]
The isomorphism $J$ between $(\mathbb{R}^n, \otimes)$ and $(\mathbb{A}_n, [,])$ allows to present an analogous product on $\mathbb{R}^m$, where $m$ is the boy $\mathbb{A}_n$. As we know the dimension of $\mathbb{A}_n$ is $\frac{n(n-1)}{2}$. So we have the product $\otimes_m$ as

\[
\otimes_m = \frac{\mathbb{R}^m \times \mathbb{R}^m}{O_m} \xrightarrow{\mathbb{A}_n \times \mathbb{A}_n} \xrightarrow{J^{-1}} (J(x), J(y))
\]

$\otimes_m$ has the properties of vector-product. That is

1) $\otimes_m$ is anti-symmetric
2) For every $X, Y \in \mathbb{R}^m$, $\langle X \otimes Y, X \rangle = 0$ and $\langle X \otimes_m Y, Y \rangle = 0$
3) The product $\otimes_m$ is bi-linear.

4. The special case for $n = 3$.

Now we will show that the product we defined in definition 1 is the vector product in $\mathbb{R}^3$ well known. Consider the set $M = \{1, 2, 3\}$. Let $S_e(3)$ be the all even permutations of $M$, i.e.

\[
S_e(3) = \{(1,2,3), (2,3,1), (3,1,2)\}.
\]

Moreover $S_e(3)$ is the set all of permutations which has the properties $\sigma(2) = \tau(2), \sigma(3) = \tau(3)$ for every $\sigma, \tau \in S_e(3)$. So, For $X = (x_1, x_2, x_3), Y = (y_1, y_2, y_3) \in \mathbb{R}^3$ and $\sigma \in S_e(3)$ we have

\[
X \otimes Y = \Sigma_{\sigma} \det (X, Y, e_{\sigma(1)}) e_{\sigma(2)} \otimes e_{\sigma(3)} \ldots.
\]

If we set $e_1 = e_1$, $e_2 = -e_3$, $e_3 = -e_5$ from tableau 1. then we can give the vector product as in tableau 2.

<table>
<thead>
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<td>$e_2$</td>
<td>$-e_1$</td>
<td>0</td>
</tr>
</tbody>
</table>

Tableau 2.
Clearly, we have

\[ X \otimes Y = \det (X,Y,e_1)e_1 + \det (X,Y,e_2)e_2 + \det (X,Y,e_3)e_3 \]

\[ = \sum_{i=1}^{\frac{n(n-1)}{2}} \det (X,Y,e_i) e_i \]

ÖZET

Bu çalışmada, \( R^6 \) uzayında, \( R^3 \) teki vektörel çarpımın bir benzeri tanımlanı. Tanımlanan bu vektörel çarpımın \( R^6 \) uzayının bir Lie cebiri yaptığı gösterdik. Ayrıca \( (R^6, \otimes) \) Lie cebiri ile \( (A_4, [,]) \) Lie cebirinin izomorfik oldukları ispatlandı. Bir genelleme olarak, \( m = \frac{n(n-1)}{2} \) boyutlu uzaylarda

\[ [,] : A_n \times A_n \rightarrow A_n \]

çarpımı yardımcıyla, \( R^m \) uzayında bir genel vektörel çarpımın,

\[ R^m \times R^m \rightarrow R^m, \quad X \otimes Y = J^{-1} [J(X), J(Y)] \]

ile verilebileceği gösterildi.

REFERENCES

