THE 3− PLANE AND THE LIGHT CONE

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ABSTRACT

In this paper we show that the 3-plane passing through the origin in a space-time will intersect the light cone in two perpendicular 2-planes.

1− The Principal Planes:

In this section we will give a sketch of how the principal planes can be obtained in order to be able to discuss the way in which a 3− plane intersect the light cone in space time.

A principal plane is a diametral plane which is at right angles to the chords which it bisects. Now if the axes are rectangular, the diametral plane (whose equation is

\[ t \frac{\partial F}{\partial x} + m \frac{\partial F}{\partial y} + n \frac{\partial F}{\partial z} = 0, \text{ where} \]

\[ F(x,y,z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d = 0 \]

or \( x(a\tau + hm + gn) + y(h\tau + bm + fn) + z(g\tau + fm + cn) = 0 \)

is at right angles to the line \( \frac{x}{t} = \frac{y}{m} = \frac{z}{n} \), if

\[ \frac{a\tau + hm + gn}{t} = \frac{h\tau + bm + fn}{m} = \frac{g\tau + fm + cn}{n} \]

If each of these ratios is equal to \( \lambda \), then

\[ (a - \lambda)\tau + hm + gn = 0, \]

\[ h\tau + (b - \lambda)m + fn = 0, \]  \hspace{1cm} (i)

\[ g\tau + fm + (c - \lambda)n = 0 \]
Therefore, $\lambda$ is a root of the equation:

$$\begin{vmatrix}
    a - \lambda & h & g \\
    h & b - \lambda & f \\
    g & f & c - \lambda
  \end{vmatrix} = 0$$

or equivalently,

$$\lambda^3 - \lambda^2 (a+b+c) + \lambda(bc+ca+ab-h^2+g^2+f^2) - D = 0 \quad (\text{ii})$$

where

$$D \equiv \begin{vmatrix}
    a & h & g \\
    h & b & f \\
    g & f & c
  \end{vmatrix}$$

Equation (ii) is called the discriminating cubic. It gives three values of $\lambda$, to each of which corresponds a set of values for $(t, m, n)$ and by substituting these sets in the equation $\frac{\partial F}{\partial x} + m \frac{\partial F}{\partial y} + n \frac{\partial F}{\partial z} = 0$, which by means of the relations (i) reduces to $\lambda (tx+my+nz) = 0$, we obtain the equations of three principal planes [1].

2- The Main Result:

A 3-plane in space time has the equation $\Sigma A_r x_r + B = 0$, $r = 1, 2, 3, 4$. This equation will reduce to

$$\Sigma A_r x_r = 0, \ r = 1, 2, 3, 4, \quad (1)$$

if the 3-plane is passing through the origin. In this case it will have a unique orthogonal line through the origin with equations $x_r = A_r u$, where $u$ is a parameter.

Now consider the equation of the light cone:

$$\sum_{i=1}^{3} x_i^2 - x_4^2 = 0 = \langle x, x \rangle = 0 \quad (2)$$

and rewrite equation (1) as follows:

$$\sum_{i=1}^{3} B_i x_i + x_4 = 0, \ B_i = A_i / A_4 \quad (1)'$$

From (1)' and (2) we have

$$\left( \sum_{i=1}^{3} B_i x_i \right)^2 = (-x_4)^2 = x_4^2 - \sum_{i=1}^{3} x_i^2$$
\[ \sum_{i=1}^{3} B_i^2 x_i^2 + 2 \sum_{i,j=1 \atop i \neq j}^{3} B_i B_j x_i x_j = \sum_{i=1}^{3} x_i^2 \]

or

\[ \sum_{i=1}^{3} C_i^2 x_i^2 + 2 \sum_{i,j=1 \atop i \neq j}^{3} B_i B_j x_i x_j = 0, \quad C_1 = B_1^2 - 1 \quad (3) \]

The discriminating cubic for equation (3) is:

\[
\lambda^3 - \lambda^2 (C_1 + C_2 + C_3) + \lambda (C_2 C_3 + C_3 C_1 + C_1 C_2 - B_2^2 B_3^2 - B_1^2 B_3^2 - B_1^2 B_2^2) - D = 0,
\]

where

\[
D = \begin{vmatrix}
C_1 & B_1 B_2 & B_1 B_3 \\
B_2 B_1 & C_2 & B_2 B_3 \\
B_1 B_3 & B_2 B_3 & C_3 \\
\end{vmatrix} = C_1 + C_2^2 + C_3 + 2 \neq 0
\]

Thus: \( \lambda^3 - A \lambda^2 - (2A + 3) \lambda - (A + 2) = 0, \) \( A \equiv C_1 + C_2^2 + C_3, \) and so, \( (\lambda + 1)^2 [\lambda - (A + 2)] = 0. \) It follows that \( \lambda = -1, -1, A+2. \)

Consider first \( \lambda = -1. \) In this case the set of equations (i) may reduce to the single equation:

\[ B_1 t_1 + B_2 m_1 + B_3 n_1 = 0, \quad i = 1, 2, 3, \quad (4) \]

If we consider \( \lambda = A + 2, \) the set of equations (i) may take the from:

\[
(B_2^2 + B_3^2) t_3 + B_1 B_2 m_3 + B_1 B_3 n_3 = 0, \\
B_1 B_2 t_3 - (B_1^2 + B_2^2) m_3 + B_2 B_3 n_3 = 0, \\
B_1 B_3 t_3 + B_2 B_3 m_3 - (B_1^2 + B_2^2) n_3 = 0
\]

(5)

Dividing the first equation of (5) by \( B_1 \) and the second by \( B_2 \) then subtracting, we have:

\[
\frac{t_3}{B_1} = \frac{m_3}{B_2}
\]

Again from the second and third equations of (5), we get

\[
\frac{m_3}{B_2} = \frac{n_3}{B_3}
\]

Thus,
\[
\frac{\nu_3}{B_1} = \frac{m_3}{B_2} + \frac{n_3}{B_3}
\]  

(6)

From the above results we find that the single equation (4) corresponding to \(\lambda_1 = \lambda_2 = -1\), is the condition that the directions given by \((\nu_1, m_1, n_1)\) and \((\nu_3, m_3, n_3)\) should be at right angles. The principal planes corresponding to the directions \((\nu_1, m_1, n_1)\) and \((\nu_3, m_3, n_3)\) are respectively: \(\nu_1 x_1 + m_1 x_2 + n_1 x_3 = 0\) and \(\nu_3 x_1 + m_3 x_2 + n_3 x_3 = 0\), or equivalently: \(\nu_1 x_1 + m_1 x_2 + n_1 x_3 = 0\) and \(B_1 x_1 + B_2 x_2 + B_3 x_3 = 0\), where \(B_1 \nu_1 + B_2 m_1 + B_3 n_1 = 0\). It follows that the 3-plane which pass through the origin will intersect the light cone in two perpendicular planes.

REFERENCE


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