RICCI CURVATURE TENSOR OF (K+1)-RULED SURFACE IN $E^n$.

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ABSTRACT

If we choose a natural companion basis (naturliche begleithbasis) for $(k+1)$-ruled surface in the Euclidean space $E^n$, then the metric coefficients are $g_\nu_\mu = \delta_\nu_\mu$, $1 \leq \nu, \mu \leq k$.

The Ricci curvature tensor $S$ for a manifold is defined by

$$S(X,Y) = \sum R(e_i, X, Y, e_i).$$

In this paper we show that the Ricci curvature tensor of a $(k+1)$ - ruled surface in $E^n$ is

$$S = \sum_\nu_\mu \left[ g_\nu_\rho R^\sigma_\rho_\mu_\nu_\sigma + \sum_i R^i_\mu_\nu_\sigma + g_\rho_\sigma R^\nu_i_\rho_\mu_\nu + g_\rho_\sigma R^\nu_i_\mu_\rho_\nu \right] \theta_\nu \otimes \theta_\mu.$$

Here, $\{\theta_\nu\}$ is the dual basis of the local coordinate basis $\{e_\nu\}$.

I. INTRODUCTION

$(k+1)$-dimensional ruled surfaces in $E^n$ are studied by H. Frank and O. Gierig, [1], [2]. Several properties of two-dimensional ruled surfaces are also given by C. Thas, [3]. The purpose of this paper is to calculate the Ricci curvature tensor of the $(k+1)$-ruled surfaces in terms of metric coefficients $g_\nu_\mu$’s and $\theta_0, \theta_1, \ldots, \theta_k$ 1-forms where $\{\theta_i\}$ is the dual of the coordinate frame field $\{e_0, e_1, \ldots, e_n\}$.

2. FUNDAMENTAL CONCEPTS

Let the orthonormal field system $\{e_1(t), \ldots, e_k(t)\}$ defined at a point of the curve

$$\eta : I \rightarrow E^n$$

$$\eta : t \rightarrow \eta(t)$$

in $E^n$, be given. Let us now define

$$M = \bigcup_{t \in I} E_k(t)$$
where \( \text{Sp} \{ e_1(t), \ldots, e_k(t) \} = E_k(t) \). It is known that \( M \) is a submanifold of \((k+1)\)-dimension in \( E^n \).

\[
\varphi(t, u_1, \ldots, u_k) = \gamma(t) + \sum_{v=1}^{k} u_v e_v(t) \tag{2.1}
\]

is a parameterization for \( M \). \( E_k(t) \) is called the generating space of \( M \) at the point \( \gamma(t) \) and \( M \) is called a ruled surface \[. \] The vector subspace

\[ \text{Sp} \{ e_1(t), \ldots, e_k(t), \dot{e}_1(t), \ldots, \dot{e}_k(t) \} \]

is called the asymptotic bundle of \( M \) in \( E_k(t) \), and it is denoted by \( A(t) \). We have

\[ \text{dim} A(t) = k + m, \quad 0 \leq m \leq k. \]

There exists an orthonormal basis of \( A(t) \) which we denote as follows

\[ \{ e_1, e_1, \ldots, e_k, a_{k+1}, \ldots, a_{k+m} \} . \tag{2.2} \]

We may also write

\[ \dot{e}_v = \sum_{\mu=1}^{k} \alpha_{v \mu} e_\mu + \sum_{t=1}^{m} \sigma_{vt} a_{k+t} \tag{2.3} \]

Since \( \langle e_v, e_v \rangle = \delta_{v \mu}, \alpha_{v \mu} = -\alpha_{v \mu} \). We can find a basis \( \{ e_1(t), \ldots, e_k(t) \} \) of the space \( E_k(t) \) such that

\[ \dot{e}_v = \sum_{v=1}^{k} \alpha_{v \mu} e_\mu + \chi_v a_{k+v}, \quad (1 \leq v \leq m), \quad (\chi_1 \geq \ldots \geq \chi_m > 0) \]

and

\[ \dot{e}_v = \sum_{v=1}^{k} \alpha_{v \mu} e_\mu, \quad (m < v \leq k), \]

[1]. The basis \( \{ e_1(t), \ldots, e_k(t) \} \) is said to be the natural companion basis (natürliche Begleitbasis) of \( E_k(t) \). Let \( P = \varphi(t, u_1, \ldots, u_k) \in M \). The set

\[ \{ \dot{\gamma}(t) + \sum_{v=1}^{k} u_v \dot{e}_v, e_1(t), \ldots, e_k(t) \} \]

is a basis of the tangent space at the point \( P \). We can define any point \( P \) of \( E_k(t) \) by changing \( u_1, u_2, \ldots, u_k \) for a fixed value of \( t \). The space

\[ \text{Sp} \{ \dot{\gamma}, \dot{e}_1, \dot{e}_2, \ldots, \dot{e}_k, e_1, \ldots, e_k \} \]
includes the union of all the tangent spaces of $E_k(t)$ at a point $P$. This space is denoted by $T(t)$ and called the tangential bundle of $M$ in $E_k(t)$. It can be easily seen that

$$k+m \leq \dim T(t) \leq k+m+1.$$ 

If $\dim T(t) = k+m$,

$$\dot{\eta}(t) = \sum_{\nu=1}^{k} \xi_{\nu} e_{\nu} + \sum_{i=1}^{m} \eta_i a_{k+i}.$$  \hspace{1cm} (2.4)

Any base curve $p(t)$ can be written in terms of $\eta(t)$ as

$$p(t) = \eta(t) + \sum_{\nu=1}^{k} u_\nu(t) e_\nu(t)$$ \hspace{1cm} (2.5)

Then we obtain

$$\dot{p}(t) = \dot{\eta}(t) + \sum_{\nu=1}^{k} [\dot{u}_\nu(t)e_\nu(t) + u_\nu(t) \dot{e}_\nu(t)].$$

Using (2.4) we find

$$\dot{p}(t) = \sum_{\nu=1}^{k} (\xi_{\nu} + \dot{u}_\nu + \sum_{\mu} u_{\mu} \xi_{\mu,\nu}) e_{\nu} + \sum_{i=1}^{m} (\eta_i u_{i} + \eta_i) a_{k+i}.$$ \hspace{1cm} (2.6)

If the point $p(t)$ satisfy

$$\dot{e}_\nu = 0, \quad (i=1,\ldots,m),$$ \hspace{1cm} (2.7)

then the vector $\dot{p}(t)$ is in the space $E_k(t)$. As it is known the coefficients $\dot{e}_\nu, \ldots, \dot{e}_m$ are different from zero. Therefore, the scalars $u_1, u_2, \ldots, u_m$ can be uniquely from the linear system (2.7). The $k$-m variables can be chosen arbitrarily. So the set of the points $p(t)$ satisfying the system (2.7) for fixed $t$, construct a $(k-m)$-dimensional vector subspace $K_{k-m}(t)$ of $E_k(t)$.

If $\dim T(t) = k+m+1$, then

$$\dot{\eta}(t) = \sum_{\nu=1}^{m} \xi_{\nu} e_{\nu} + \sum_{i=1}^{m} \eta_i a_{k+i} + \eta_{m+1} a_{k+m+1}.$$ \hspace{1cm} (2.8)

In this case

$$\dot{p}(t) = \sum_{\nu} (\dot{u}_\nu + \sum_{\mu} \xi_{\mu,\nu} u_\nu + \xi_{\nu}) e_{\nu} + \sum_{i=1}^{m} (\eta_i u_{i} + \eta_i) a_{k+i} + \eta_{m+1} a_{k+m+1}.$$ \hspace{1cm} (2.9)
The \((k\cdot m)\)-dimensional subspace \(Z_{k\cdot m}(t)\) defined by the linear system
\[
x_q u_l + \eta_l = 0, \quad (l=1,\ldots,m)
\]
(2.10)
is said to be the central space of \(M\) in \(E_k(t)\).

**Theorem 2.1:** Let the metric coefficients of the \((k+1)\)-dimensional ruled surface in \(E^n\) be \(g_{\nu\mu}\). Then
\[
g_{00} = \sum_{\nu=1}^{k} (\zeta_{\nu} + \sum_{\nu=1}^{k} \alpha_{\nu\mu} u_{\nu})^2 + \sum_{\nu=1}^{m} (\eta_{\nu} + \zeta_{\nu} u_{\nu})^2 + (\eta_{k+1})^2
\]
(2.11)
\[
g_{0\nu} = \zeta_{\nu} + \sum_{\nu=1}^{k} \alpha_{\nu\mu} u_{\nu}
\]
\[
g_{\nu\mu} = \delta_{\nu\mu}, \quad (\nu,\nu = 1,\ldots,k)
\]
[1].

**Theorem 2.2:** Let the dual of frame field \(\{e_o, e_1, \ldots, e_k\}\) be \(\{\theta_o, \theta_1, \ldots, \theta_k\}\) where \(\{e_1, e_2, \ldots, e_k\}\) is the natural companion basis of \(E_k(t)\) and
\[
e_o = \varphi_* \left( \frac{\partial}{\partial t} \right). \quad \text{Then, the first fundamental form of } M \text{ is}
\]
\[
I = g_{00} \theta_o \otimes \theta_o + \sum_{\nu=1}^{k} g_{0\nu} (\theta_\nu \otimes \theta_o + \theta_o \otimes \theta_\nu) + \sum_{\nu=1}^{k} \theta_\nu \otimes \theta_\nu. \tag{2.12}
\]

3. **RICCI CURVATURE TENSOR OF \((K+1)\)-RULED SURFACE**

The Ricci curvature of a manifold \(M\) is the tensor field \(S\) which is defined by
\[
S(X_p, Y_p) = \sum_i R(e_{ip}, X_p, Y_p, e_{ip}) \tag{3.1}
\]
[4]. Since,
\[
R(e_{ip}, X_p, Y_p, e_{ip}) = < R(e_{ip}, X_p), Y_p, e_{ip} > \tag{3.2}
\]
then
\[
S(X_p, Y_p) = \sum < R(e_{ip}, X_p), Y_p, e_p >. \tag{3.3}
\]
We have
\[
R(e_k, e_l)e_i = \sum_j R_{ijkl}e_j. \tag{3.4}
\]
Theorem 3.1: The Ricci curvature tensor of \((k+1)\)-Ruled surface is

\[
S = \sum_{\nu, \nu=0}^{k} [R_{\mu
u\nu\delta}^{\nu} + \sum_{i=1}^{k} (R_{i\mu\nu\nu}^{\nu} + g_{10} (R_{i\mu\nu}^{\nu} + R_{i\mu\nu}^{\nu}))] \theta_{\nu} \otimes \theta_{\mu}. \tag{3.5}
\]

Proof. Let \(X = \sum_{\nu=0}^{k} x_{\nu}e_{\nu}, \ Y = \sum_{\nu=0}^{k} y_{\nu}e_{\nu}\). Since \(R\) is a tensor field, we have

\[
R(e_{i}, X) Y = R(e_{i}, \sum_{\nu=0}^{k} x_{\nu}e_{\nu}) (\sum_{\nu=0}^{k} y_{\nu}e_{\nu}) = \sum_{\nu, \mu=0}^{k} x_{\nu}y_{\nu}R(e_{i}, e_{\nu})e_{\mu}.
\]

By the equation (3.4), we find

\[
R (e_{i}, X) Y = \sum_{\nu, \mu, h}^{k} x_{\nu}y_{\mu} R_{h\mu\nu\nu}^{h} e_{h}. \tag{3.6}
\]

From (3.2), we obtain

\[
S (X, Y) = \sum_{i=0}^{k} \sum_{\nu, \mu, h} x_{\nu}y_{\mu} R_{h\mu\nu\nu}^{h} e_{h}, e_{i} >
\]

\[
= \sum_{\nu, \mu} x_{\nu}y_{\mu} R_{\mu\nu\nu\nu}^{h} < e_{h}, e_{i} >
\]

\[
= \sum_{\nu, \mu} x_{\nu}y_{\mu} (\sum_{i, h} R_{h\mu\nu\nu}^{h} < e_{h}, e_{i} >).
\]

Since \(< e_{h}, e_{i} > = g_{ih}\), using (2.11), we find

\[
S (X, Y) = \sum_{\nu, \mu} x_{\nu}y_{\mu} (\sum_{i, h} R_{h\mu\nu\nu}^{h} g_{ih})
\]

\[
= \sum_{\nu, \mu} x_{\nu}y_{\mu} [R_{\mu\nu\nu\nu}^{h} + \sum_{i=1}^{k} R_{i\mu\nu\nu}^{h} + \sum_{i=1}^{k} g_{i0} (R_{i\mu\nu}^{h} + R_{i\mu\nu}^{h})]
\]

\[
= \sum_{\nu, \mu=0}^{k} x_{\nu}y_{\mu} [R_{\mu\nu\nu\nu}^{h} + \sum_{i=1}^{k} (R_{i\mu\nu}^{h} + g_{i0} (R_{i\mu\nu}^{h} + R_{i\mu\nu}^{h})].
\]

Let \(\{\theta_{0}, \theta_{1}, ..., \theta_{k}\}\) be the dual of the basis \(\{e_{0}, e_{1}, ..., e_{k}\}\). Since

\((\theta_{\nu} \otimes \theta_{\nu}) (X, Y) = \theta_{\nu} (X) \quad \theta_{\mu} (Y) = x_{\nu}y_{\mu},\)
we have (3.5).

If we calculate the Christoffel symbols $\Gamma^l_{jk}$ for a $(k+1)$-ruled surface, we find

$$\Gamma^0_{00} = \frac{1}{2g} \left[ \frac{\partial g}{\partial s} + \sum_{\nu=1}^{k} \left( \zeta_{\nu} + \sum_{\mu=1}^{k} x_{\nu\mu} u_\mu \right) \frac{\partial g}{\partial u_\nu} \right]$$

$$\Gamma^\nu_{00} = \frac{1}{2g} \left[ -\left( \zeta_\lambda + \sum_{\mu} x_{\lambda\mu} u_\mu \right) \frac{\partial g}{\partial t} + \sum_{\nu} \left( \zeta_\nu + \sum_{\mu} x_{\nu\mu} u_\mu \right) \frac{\partial g}{\partial u_\nu} \right] + 2g \left( \zeta_\lambda + \sum_{\nu} x_{\lambda\nu} u_\mu + \sum_{\nu} \left( \zeta_\nu + \sum_{\mu} x_{\nu\mu} u_\mu \right) x_{\lambda\nu} - \frac{1}{2} \frac{\partial g}{\partial u_\lambda} \right)$$

$$\Gamma^0_{\nu\mu} = \Gamma^\nu_{0\mu} = 0, \; (1 \leq \lambda, \nu, \mu \leq k)$$

$$\Gamma^0_{\nu\lambda} = \Gamma^\nu_{0\lambda} = \frac{1}{2g} \frac{\partial g}{\partial u_\lambda}$$

$$\Gamma^\nu_{\lambda\nu} = \Gamma^\nu_{0\lambda} = \frac{1}{2g} \left[ -\left( \zeta_\nu + \sum_{\mu} x_{\nu\mu} u_\mu \right) \frac{\partial g}{\partial u_\nu} + 2g x_{\nu\lambda} \right].$$

So, the Ricci curvature of the Ruled surface can be given in terms of the functions $x_{ij}$ and metric coefficients of the surface.

REFERENCES


