Almost Regularity Of Dual Summability Methods

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SUMMARY

In this paper, we have defined almost regularity for a series method and investigated necessary and sufficient conditions for a series method to be almost regular. We also examined the almost regularity for dual summability methods.

1. INTRODUCTION

Let \( l_{\infty}, c, c_0 \) and \( \gamma \) be the linear space of all bounded sequences, convergent sequences, null sequences and convergent series with complex terms, respectively. A sequence \( s = (s_n) \in l_{\infty} \) is almost convergent if and only if

\[
\lim_{p \to \infty} \frac{1}{p} \sum_{v=n}^{n+p-1} s_v = b
\]

uniformly in \( n \). [4]. This is written as \( \text{f-lims} = b \). We denote by \( F \) the space of all almost convergent sequences and by \( F_0 \) the subspace of \( F \) consisting of all sequences almost convergent to zero. If a sequence \( s = (s_n) \) is convergent and its limit \( b \), then the sequence \( (s_n) \) is almost convergent to \( b \), but not conversely in general.

Now let \( A = (a_{nk}) \) be an infinite matrix with complex terms. If the series

\[
(1) \quad t_n = \sum_{k=1}^{\infty} a_{nk} s_k
\]
converges for \( n = 1, 2, \ldots \), then \( A(s) = (t_n) \) is said to be the A-transform of \( s \). Hence, the summability method \( A \) is sequence method.

The matrix \( A = (a_{nk}) \) is called regular if the A-transform of \( s \) is convergent to the limit of \( s \) for each \( s \in c \).

A sequence \( s \) is said to be almost A-summable if the A-transform of \( s \) is almost convergent.

The matrix \( A = (a_{nk}) \) is called almost regular if the sequence \( A(s) \) almost almost converges to the limit of \( s \) for each \( s \in c \).

Throughout the paper, the sums will be taken from 1 to \( \infty \).

The following theorem is due to King, J.P., [2].

THEOREM 1.1. The matrix \( A = (a_{nk}) \) is almost regular if and only if

(i) \( \|A\| = \sup_n \sum_k |a_{nk}| < \infty \)

(ii) \( f\text{-}\lim a_{nk} = 0 \), for each \( k \)

(iii) \( f\text{-}\lim \sum_{k=1}^{\infty} a_{nk} = 1 \).

The following result may easily be obtained:

COROLLARY 1.2. A matrix \( A = (a_{nk}) \) transforms \( c_0 \) into \( F_0 \) (i.e., \( A \in (c_0, F_0) \)) if and only if

(i) \( \|A\| = \sup_n \sum_k |a_{nk}| < \infty \)

(ii) \( f\text{-}\lim a_{nk} = 0 \), for each \( k \).

2. DUAL SUMMABILITY METHODS.

Let \( A = (a_{nk}) \) be a sequence method given by (1). Suppose that, for each \( n \), the series

\[ \sum_{k=1}^{\infty} a_{nk} \]

converges; this is a much weaker assumption than the regularity of \( A \). Then we define
\[ b_{nk} = \sum_{i=k}^{\infty} a_{ni}. \]

Further, suppose that the sequence \( (s_k) \) is defined by
\[ s_k = \sum_{i=1}^{k} x_i. \]

Let \( B \) denote the summability method given by the series-to-sequence transformation (series method)
\[ v_n = \sum_{k=1}^{\infty} b_{nk} x_k, \quad (n=1,2,\ldots). \]

The methods \( A \) and \( B \) are called dual summability methods, \([3]\). It is well-known that \( A \) is regular if and only if \( B \) is regular, \([1]\).

3. MAIN RESULTS

The strong regularity of a sequence method has been defined by Lorentz, G.G., \([4]\).

Recently, Öztürk, E., \([5]\), have defined strong regularity of a series method and showed that \( A \) is strongly regular if and only if \( B \) is strongly regular, where \( A \) and \( B \) are two dual summability methods.

As we mentioned in Section 1, the almost regularity of a sequence method has been introduced by King, J.P., \([2]\).

Similarly, we shall define almost regularity for series method and also investigate almost regularity of dual summability methods.

Let us suppose throughout that the method \( B \) is a series method.

DEFINITION. 3.1. Let \( \sum x_k \) be a convergent series. If the method \( B \) almost sums the series \( \sum x_k \), i.e., if f-lim \( B(x) \) exists, then \( B \) is said to be almost conservative. Moreover, if f-lim \( B(x) = \sum x_k \)

whenever \( x=(x_k) \in \gamma \), then \( B \) is called almost regular.

Now we can give the following
THEOREM 3.2. A matrix $B = (b_{nk})$ is almost regular if and only if

i) $\sup_n \sum_k |\Delta b_{nk}| < \infty$

ii) $f$-lim $b_{nk} = 1$, for all $k$,

where $\Delta b_{nk} = b_{nk} - b_{n,k+1}$.

Proof. Let us suppose that $B$ is almost regular. Then, for every $x \in \gamma$

$$B_n (x) = \sum_k b_{nk} x_k$$

converges, for each $n$, and $f$-lim $B_n (x) = \sum_k x_k$. If we now put $x = e_k$,

$(k = 1, 2, \ldots)$, then the necessity of (ii) is trivial, where $e_k$ is a sequence

whose $k$-th component is one and the others are zero.

Now, by Abel's partial summation, we get

$$\sum_{k=1}^{m} b_{nk} x_k = \sum_{k=1}^{m-1} \Delta b_{nk} (s_k - a) + a b_{n1} + (s_m - a) b_{nm}$$

for each $n$, where $(x_k) \in \gamma$ and $s_m$ is the $m$-th partial sum of the series

$$\sum_k x_k$$

and $\sum_k x_k = a$. On the other hand, it is well-known that, if

$$\sum_k b_{nk} x_k$$

converges for each $n$ whenever $x \in \gamma$, then

$$\sup_k |b_{nk}| < \infty, \text{ (for each } n)$$

Hence, letting $m \to \infty$ in (3), we get

$$\sum_k b_{nk} x_k = \sum_k \Delta b_{nk} (s_k - a) + a b_{n1}.$$

Since $f$-lim $B (x) = a$ exists and $f$-lim $b_{n1} = 1$, then we must have

$$\sum_k \Delta b_{nk} (s_k - a) \in F_o.$$

Therefore, one can easily see that (i) is necessary since $(s_k - a) \in c_o$.

Conversely, suppose now that the condition (i) and (ii) hold. From

(i), the series $\sum_k (b_{nk} - b_{n,k+1})$ is convergent and $m \lim b_{nm}$ exists for

every $n$. So the statement (4) holds for each $x \in \gamma$ with $\sum_k x_k = a$. By (i)

and (ii), it is easily seen that

$$C = (\Delta b_{nk}) \in (c_0, F_o)$$
since the conditions of Corollary 1.2 are satisfied. Hence \( f\)-\( \lim \) \( B(x) = a \) is obtained. Thus the proof is completed.

**THEOREM 3.3.** Let \( A \) and \( B \) be two dual summability methods. Then \( A \) is almost regular if and only if \( B \) is almost regular.

**Proof.** Since

\[
\Delta b_{nk} = b_{nk} - b_{n-k+1} = a_{nk}, \text{ for all } n,k. \text{ Hence,}
\]

\[
\sup_n \sum_k |a_{nk}| < \infty \text{ if and only if } \sup_n \sum_k |\Delta b_{nk}| < \infty. \text{ Further,}
\]

\[
f\text{-}\lim \sum_k a_{nk} = 1 \text{ and } f\text{-}\lim a_{nk} = 0, \text{ for each } k, \text{ if and only if}
\]

\[
f\text{-}\lim b_{nk} = 1, \text{ (for each } k). \text{ Now the proof follows from Theorem 1.1 and Theorem 3.2.}
\]

**ÖZET**

Bu çalışmada, bir seri metodunun hemen hemen regülerliğini tanımladık ve bir seri metodunun hemen hemen regüler olması için gerek ve yeter şartları verdik. Ayrıca, dual toplanabilirme metodlarının hemen hemen regülerliğini araştırdık.

**REFERENCES**


