Fixed Point Theorems

by

M.S. KHAN and M.D. KHAN

Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie
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M.S. KHAN and M.D. KHAN*

(Received October: 4, 1983, accepted January 30, 1984)

Department of Mathematics Aligarh Muslim University ALIGARH- 202001, INDIA.
* Central School, Babina Cantt-204401, INDIA.

SUMMARY

Some sufficient conditions have been obtained for self mappings of a complete metric space to have a unique fixed point. Our work generalizes some known results.

I. INTRODUCTION

Let \((X,d)\) be a metric space. A mapping \(T:X\rightarrow X\) is called a contraction mapping if there exists a real number \(\alpha\), \(0 \leq \alpha < 1\) such that

\[
(A) \quad d(Tx, Ty) \leq \alpha \, d(x, y), \text{ for all } x, y \in X.
\]

The Banach contraction principle states that a contraction mapping on a complete metric space has a unique fixed point. Kannan [1] proved the following result.

**Theorem.** If \(T\) is a mapping of a complete space \(X\) into itself such that

\[
(B) \quad d(Tx, Ty) \leq \alpha \{d(x, Tx) + d(y, Ty)\},
\]

for all \(x, y \in X\), \(0 \leq \alpha < \frac{1}{2}\), then \(T\) has a unique fixed point.

Further, Kannan [2] gave an example to show that conditions (A) and (B) are independent. In this paper, we have obtained some fixed point theorems for mappings satisfying conditions more general than those of Banach and Kannan.

II. RESULTS

**Theorem 1.** If \(T\) is a self mapping of a complete metric space \((X, d)\) and if for some positive integer \(p\), the following inequality

\[
\text{Theorem 1.}
\]
(C) \[ d(T^{2p}x, T^{2p}y) \leq a_1 d(T^p x, T^{2p}x) + a_2 d(T^p y, T^{2p}y) + a_3 d(T^p x, T^p y), \]
holds for all \( x, y \in X, a_1 \geq 0, a_2 \geq 0, a_3 \geq 0 \) and \( a_1 + a_2 + a_3 < 1 \), then \( T \) has a unique fixed point provided \( T^p \) is continuous.

**Proof.** Let \( x \in X \). Set \( T^n(x) = x_0 \) and \( T^n(x_{n-1}) = x_n \).

Also suppose \[ K = \frac{a_1 + a_3}{1 - a_2}. \]

Then,
\[ d(x_1, x_2) = d(T^{2p}x, T^{2p}x_0) \leq a_1 d(T^p x, T^{2p}x) + a_2 d(T^p x_0, T^{2p}x_0) + a_3 d(T^p x, T^p x_0) \leq a_1 d(x_0, x_1) + a_2 d(x_1, x_2) + a_3 d(x_0, x_1). \]

Hence \[ d(x_1, x_2) \leq K d(x_0, x_1). \]

Again,
\[ d(x_2, x_3) = d(T^{2p}x_0, T^{2p}x_1) \leq a_1 d(T^p x_0, T^{2p}x_0) + a_2 d(T^p x_1, T^{2p}x_1) + a_3 d(T^p x_0, T^p x_1), \leq a_1 d(x_1, x_2) + a_2 d(x_2, x_3) + a_3 d(x_1, x_2). \]

So \[ d(x_2, x_3) \leq K^2 d(x_0, x_1). \]

In general, we have \[ d(x_n, x_{n+1}) \leq K^n d(x_0, x_1). \]

Thus \( \{x_n\} \) is a Cauchy sequence which converges to some \( w \in X \). Now \[ d(w, T^{2p}w) \leq d(w, x_{n+2}) + d(T^{2p}x_n, T^{2p}w), \leq d(w, x_{n+2}) + a_1 d(T^p x_n, T^{2p}x_n) + a_2 d(T^p w, T^{2p}w) + a_2 d(T^p x_n, T^p w), \leq d(w, x_{n+2}) + a_1 d(x_{n+1}, x_{n+2}) + a_2 d(T^p w, x_{n+1}) + a_2 d(x_{n+1}, x_{n+2}) + a_2 d(x_{n+2}, T^{2p}w) + a_3 d(x_{n+1}, T^p w). \]

The right hand side can be made arbitrarily small by choosing \( n \) sufficiently large.

Hence \[ T^{2p}(w) = w. \]
For uniqueness of \( w \), let \( w^* \) be another fixed point of \( T^{2p} \). Then
\[
\begin{align*}
  d(w,w^*) &= d(T^{4p}w, T^{4p}w^*) \\
  &\leq a_1 d(T^{2p}w, T^{2p}w) \\
  &+ a_2 d(T^{4p}w^*, T^{4p}w^*) \\
  &= a_1 d(T^{4p}w, T^{2p}w) + a_2 d(T^{4p}w^*, T^{2p}w^*) + a_3 d(T^{4p}w, T^{4p}w^*).
\end{align*}
\]
So
\[
(1 - a_3) d(T^{4p}w, T^{4p}w^*) = 0.
\]

Hence \( w \) is the unique fixed point of \( T^{2p} \), and therefore \( w \) is the unique fixed point of \( T \).

**Remark.** Theorem 1 can be extended by replacing condition (C) by any one the following conditions:
\[
\begin{align*}
(D_1) \quad d(T^{2p}x, T^{2p}y) &\leq a_1 d(T^p x, T^{2p}x) + a_2 d(T^p y, T^{2p}y) \\
  &+ a_3 d(T^p x, T^{2p}y) + a_4 d(T^p y, T^{2p}x) + a_5 d(T^p x, T^p y),
\end{align*}
\]
for all \( x,y \in X \), \( a_i \geq 0 \), \( 1 \leq i \leq 5 \), \( \sum_{i=1}^{5} a_i < 1 \).

\[
(D_2) \quad d(T^{2p}x, T^{2p}y) \leq \alpha \max \{ d(T^p x, T^{2p}x), d(T^p y, T^{2p}y), d(T^p x, T^p y) \}
\]
where \( \alpha \in [0,1) \).

**Example.** Now we give an example where \( T \) does not satisfy conditions (A) and (B) but satisfies condition (C).

Let \( X = [0,1] \) with the usual metric. Suppose \( T : X \to X \), be defined by
\[
T(0) = T(1) = 0,
T(x) = 1 \text{ for } x \in (0,1).
\]
Then
\[
T^{2}(x) = T(1) = 0 \text{ for all } x \in [0,1].
\]
Hence \( T^{2} \) is continuous and satisfies condition (C). Since \( T \) is not continuous, it does not satisfy condition (A). Also it does not satisfy condition (B) as is evident by taking \( x = 0 \) and \( y = 1/2 \). For
\[
\begin{align*}
  d(T(0), T(\frac{1}{2})) &\leq \alpha [d(0,T(0)) + d(\frac{1}{2}, T(\frac{1}{2}))] \\
  d(0,\frac{1}{2}) &\leq \alpha d(\frac{1}{2}, 1)
\end{align*}
\]
\( 1 \leq \alpha \frac{1}{2} \), which is not admissible. We also prove the following.
Theorem 2. Let \((X, d)\) be a complete metric space and \(T\) a self mapping of \(X\) satisfying.

\[
\begin{align*}
\text{(E)} & \quad d(T^{p+1}x, T^{p+1}y) \leq a_1 d(T^p x, T^{p+1}x) + a_2 d(T^p y, T^{p+1}y) \\
& \quad + a_3 d(T^p x, T^{p+1}y) + a_4 d(T^p y, T^{p+1}x) + a_5 d(T^p x, T^p y)
\end{align*}
\]
for all \(x, y \in X\).

\[
\begin{align*}
\text{(F)} & \quad \sum_{i=1}^{5} a_i < 1, \quad a_i \geq 0, \quad 1 \leq i \leq 5,
\end{align*}
\]

\[
\begin{align*}
\text{(G)} & \quad a_1 = a_2 \text{ or } a_3 = a_4
\end{align*}
\]

\[
\begin{align*}
\text{(H)} & \quad T^p \text{ is continuous.}
\end{align*}
\]

Then \(T\) has a unique fixed point.

**Proof.** For an arbitrary \(x \in X\), define \(x_0 = T^p(x)\) and \(x_n = T^{(n-1)}(x_0)\).

Then

\[
\begin{align*}
d(x_1, x_2) &= d(T^{p+1}x, T^{p+2}x) \leq a_1 d(T^p x, T^{p+1}x) \\
& \quad + a_4 d(T^{p+1}x, T^{p+2}x) + a_5 d(T^p x, T^{p+2}x) \\
& \leq a_1 d(x_0, x_1) + a_2 d(x_1, x_2) + a_3 d(x_0, x_2) \\
& \quad + a_4 d(x_1, x_1) + a_5 d(x_0, x_1).
\end{align*}
\]

Hence

\[
\begin{align*}
d(x_1, x_2) &\leq \left( \frac{a_1 + a_3 + a_5}{1 - a_2 - a_3} \right) d(x_0, x_1) = K d(x_0, x_1)
\end{align*}
\]

where

\[
K = \left( \frac{a_1 + a_3 + a_5}{1 - a_2 - a_3} \right).
\]

Further

\[
\begin{align*}
d(x_2, x_3) &= d(T^{p+2}x, T^{p+3}x) \leq a_1 d(T^{p+1}x, T^{p+2}x) + a_2 d(T^{p+2}x, T^{p+3}x) \\
& \quad + a_3 d(T^{p+1}x, T^{p+3}x) + a_4 d(T^{p+2}x, T^{p+2}x) \\
& \quad + a_5 d(T^{p+1}x, T^{p+2}x) \\
& \leq a_1 d(x_1, x_2) + a_2 d(x_2, x_3) + a_3 d(x_1, x_3) + a_4 d(x_2, x_2) + a_5 d(x_1, x_2).
\end{align*}
\]
Hence
\[ d(x_2, x_3) \leq K \ d(x_1, x_2) \leq K^2 \ d(x_0, x_1). \]

In general,
\[ d(x_n, x_{n+1}) \leq K^n \ d(x_0, x_1). \]

Thus \( \{x_n\} \) is a Cauchy sequence which converges to some point \( w \in X \). Now,
\[
\begin{align*}
d(T^p w, T^{p+1} w) &\leq d(T^p w, x_{n+p+1}) + d(T^{p+1} x_n, T^{p+1} w) \\
&\leq d(T^p w, x_{n+p+1}) + a_1 d(T^p x_n, T^{p+1} x_n) + a_2 d(T^p w, T^{p+1} w) \\
&\quad + a_3 d(T^p x_n, T^{p+1} w) + a_4 d(T^{p+1} x_n, T^p w) + a_5 d(T^p x_n, T^p w) \\
&\leq d(T^p w, x_{n+p+1}) + a_1 d(x_{n+p}, x_{n+p+1}) + a_2 d(T^p w, w) \\
&\quad + a_3 d(x_{n+p}, T^{p+1} w) + a_3 d(x_{n+p}, T^{p+1} w) + a_3 d(w, T^{p+1} w) \\
&\quad + a_4 d(x_{n+p+1}, T^p w) + a_5 d(x_{n+p}, T^p w).
\end{align*}
\]

Hence,
\[
(1-a_2-a_3) \ d(T^p w, T^{p+1} w) \leq d(w, x_{n+p+1}) + a_1 d(x_{n+p}, x_{n+p+1}) \\
+ a_2 d(T^p \lim_{n \to \infty} x_n, w) + a_3 d(x_{n+p}, T^p \lim_{n \to \infty} x_n) \\
+ a_4 d(x_{n+p+1}, T^p \lim_{n \to \infty} x_n) + a_5 d(x_{n+p}, T^p \lim_{n \to \infty} x_n).
\]

Using the continuity of \( T^p \) and letting \( n \to \infty \), we get \( T^p(w) = T^{p+1}(w) \).

Hence \( T^p(w) \) is a fixed point of \( T \).

For the unicity of \( T^p(w) \), consider an other fixed point \( w^* \neq T^p(w) \) of \( T \).

Then
\[
\begin{align*}
d(T^p w, w^*) &= d(T^p w, T^{p+1} w^*) = d(T^{p+1} w, T^p w^*) \\
&\leq a_1 d(T^p w, T^{p+1} w) + a_2 d(T^p w^*, T^{p+1} w^*) + a_3 d(T^p w, T^{p+1} w^*) \\
&\quad + a_4 d(T^p w^*, T^{p+1} w) + a_5 d(T^p w, T^p w^*) \\
&= a_3 d(T^p w, T^{p+1} w^*) + a_4 d(T^{p+1} w^*, T^p w) \\
&\quad + a_5 d(T^p w, T^{p+1} w^*).
\end{align*}
\]

This gives \( d(T^p w, T^{p+1} w^*) = d(T^p w, w^*) = 0 \).

Therefore \( T^p(w) \) is the unique fixed point of \( T \). This completes the proof.
Remarks: (i) Theorem 2 can be extended by replacing condition (E) by any one of the following:

\[(E_1) \quad d(T^{p+1}x, T^{p+1}y) \leq \alpha \max\{d(T^px, T^{p+1}x), d(T^py, T^{p+1}y),
\]
\[d(T^px, T^{p+1}y), d(T^py, T^{p+1}x), d(T^px, T^py)\},
\]

where
\[\alpha \in [0, \frac{1}{2}).\]

\[(E_2) \quad d(T^{p+1}x, T^{p+1}y) \leq \alpha \max\{d(T^px, T^py), d(T^px, T^{p+1}x),
\]
\[d(T^py, T^{p+1}y), \frac{1}{2} [d(T^px, T^{p+1}y) + d(T^py, T^{p+1}x)]\},
\]

where
\[\alpha \in [0,1).\]

(ii) The example given in section 2 also shows that conditions (A), (B) and (E) are different and continuity of T is not necessary.

(iii) For \(p = 1\), conditions (D_1) and (E) are same.

(iv) For \(p = 0\), Theorem 2 reduces to that of Hardy and Rogers [3].

(v) In Theorem 2, if \(a_i\)'s are replaced by functions from \((0, \infty)\) to \((0, \infty)\) such that \(\lim_{t \to 0} (a_1(t) + a_4(t)) < 1\) and \(\lim_{t \to 0} (a_2(t) + a_3(t)) < 1\), we get a result of Chi Song Wong [4].

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