Banach Limits And Infinite Matrices (II)

by

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ABSTRACT

An inequality sharper than that of Knopp’s Core inequality was proved in [3]. In the present paper a generalised result of the above inequality for row finite matrices is proved with the help of a sublinear functional $\Omega_B$ Some sets which arise in connection with $\Omega_B$ are also characterised.

INTRODUCTION

Let $m$ denote the Banach space of all bounded real sequences $x = \{x_n\}_{n=1}^\infty$, normed by $\|x\| = \sup_n |x_n|$. We write

$$m_0 = \{x \in m : \sup_n \sum_{i=1}^{n} |x_i| < \infty\}.$$ 

Define $L : m \rightarrow R$ by $L(x) = \lim_n \sup x_n$. The space $c$ of all convergent real sequences is a closed subspace of $m$.

Banach limits [1] are linear functionals $G$ on the space $m$ satisfying conditions:

(i) $x \geq 0 \Rightarrow G(x) \geq 0$,
(ii) $G(e) = 1$,
(iii) $G(\sigma x) = G(x),$

where $e = (1,1,\ldots)$ and $\sigma : m \rightarrow m$ is defined by $(\sigma x_n) = x_{n+1}$. Condition (iii) is the same thing as saying that $G$ is $\sigma$-invariant on $m$ and $\sigma$ is called a shift operator. Let $\beta$ denote the set of all Banach limits on $m$. 
If $P$ is a sublinear functional on $m$, we write $\langle m, P \rangle$ to denote the set of all linear functionals $Q$ on $m$ such that $Q(x) \leq P(x)$ for all $x \in m$. A sublinear functional $P$ on $m$ *generates* Banach limits if for a linear functional $G$ on $m$, $G \leq P \Rightarrow G \in \beta$; (that is, if $\langle m, P \rangle \subset \beta$). A sublinear functional $P$ *dominates* Banach limits if $G \in \beta \Rightarrow G \leq P$; (that is if $\beta \subset \langle m, P \rangle$). Thus if $P$ both dominates and generates Banach limits then $\beta = \langle m, P \rangle$.

Let $A = (a_{nk})$ be an infinite matrix of real numbers and write $A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k$ if it converges for all $n > 0$. We then write $Ax = \{A_n(x)\}_{n=1}^{\infty}$. Note that the matrix $A$ is called *regular* if $A: c \rightarrow c$ and $\lim Ax = \lim x$. The Silverman-Toeplitz conditions for a regular matrix are the following:

(i) $\|A\| = \sup_n \sum_k |a_{nk}| < \infty$,

(ii) $\lim_{n} a_{nk} = 0$, for fixed $k$,

(iii) $\lim_{n} \sum_k a_{nk} = 1$.

A matrix $A$ is called *strongly regular* [4] if it is regular and

$$\lim_{n} \sum_k |a_{nk} - a_{n,k+1}| = 0.$$ 

We say that $A = (a_{nk})$ is *almost positive* if $\lim_{n} \sum_k a_{nk}^- = 0$ (if $\lambda \in \mathbb{R}$, $\lambda^+$ means max $(\lambda, 0)$ and $\lambda^-$ means max $(−\lambda, 0)$). If $A$ is regular, it is almost positive if and only if $\lim_{n} \sum_k |a_{nk}| = 1$ (see [7]).

The main object of this paper is to establish an inequality for a row finite matrix $A$ and for a sublinear functional defined on $m_B$ for a normal matrix $B$. This is proved in section 3, and it is a generalisation of Theorem 3 of [3] for a row finite matrix. In section 4 sets which arise in connection with $\Omega_B$ have been studied. Section 5 deals with a set of section 4 where $m_0$ is replaced by a bounded subspace $V$ of $m$.

2. Let $s$ be the set of all real sequences $x = \{X_n\}_{n=1}^{\infty}$. We write $m_A = \{x \in s: Ax \in m\}$, $m_{AO} = \{x \in s: Ax \in m_0\}$. It is evident that $m_A$
is a linear space and \( m_{A_0} \) is a subspace. Further if we define, for \( x \in m_A \),
\[
\| x \| = \sup_n \sum_k |a_{nk}x_k|
\]
then it is a seminorm on \( m_A \). It is a norm if
\( A \) is invertible. It is also familiar that
\[
A: m \rightarrow s \iff \sum_k |a_{nk}| < \infty \quad \text{(for each } n) ;
\]
\[
A: m \rightarrow m \iff \| A \| = \sup_n \sum_k |a_{nk}| < \infty .
\]
Let \( c_A \) be the summability field of \( A \); that is,
\[
c_A = \{ x \in s: L(Ax) = -L(-Ax) \} .
\]
It is evident that
\[
c_A \subset m_A \quad (2.1)
\]  
It is also easily seen that
\[
m \cap m_A = m \iff \| A \| < \infty . \quad (2.2)
\]
It is in order to quote the following theorem.

**Theorem:** (Mazur-Orlicz [6]). Let \( A \) be a regular matrix. Then \( c_A \cap c' \neq \emptyset \Rightarrow c_A \cap m' \neq \emptyset \); where \( c' \) and \( m' \) are the complementary sets of \( c \) and \( m \) respectively. In other words, if a regular matrix evaluate some divergent sequence, then it must evaluate an unbounded sequence; that is, if a regular matrix evaluates no unbounded sequence, then it evaluates only convergent sequences. From (2.2) we have \( \| A \| < \infty \Rightarrow m \subset m_A \) and there are important cases where \( m \) is a proper subset of \( m_A \). For example, if \( A \) is regular such that \( A \) evaluates some divergent sequence (infact, these cases are only important), then the above theorem gives that \( c_A \cap m' \neq \emptyset \) and therefore from (2.1) we have \( m_A \cap m' \neq \emptyset \).

Let \( w: m \rightarrow R \) be defined by
\[
w(x) = \inf_{z \in m_0} L(x+z)
\]
It is easy to see that \( w \) is a sub-linear functional. The result which was proved in [3] is the following:

**Theorem:** \( \limsup_n A_n(x) \leq w(x) (x \in m) \)

if and only if the matrix \( A \) is almost positive an strongly regular.
The following lemmas are required to prove the main theorem and the proposition.

Lemma 1: (Knopp's Core Theorem) \( L(Ax) \leq L(x) \) (\( x \in m \)) if and only if \( A \) is almost positive and regular.

Lemma 2: (Simons [7], Corollary 12, Theorem 11). If

(i) \( \sum_{k} |a_{nk}| < \infty \) (for each \( n \))

(ii) \( a_{nk} \to 0 \) (\( n \to \infty \)) for fixed \( k \),

then there exists \( y \in m: \|y\| \leq 1 \) and

\[
\lim \sup_{n} \sum_{k} a_{nk} y_{k} = \lim \sup_{n} \sum_{k} |a_{nk}|.
\]

3. Now suppose that \( \|B\| < \infty \) and we write, for any real matrix \( B \), and for \( x \in m_{B} \)

\[
\Omega_{B}(x) = \inf_{z \in m_{B_{0}}} L(B(x+z)). \tag{3.1}
\]

The function \( \Omega_{B}: m_{B} \to \mathbb{R} \) is well-defined if we suppose that

\[
\lim \sup_{n} B_{n} z \geq 0, \ (z \in m_{B_{0}}), \tag{3.2}
\]

(see Devi [3], regarding the functional \( q_{y} \) before the statement of Theorem 1).

In the case

\( b_{nk} \to 0 \) (\( n \to \infty \), \( k \) fixed)

by Abel’s transformation,

\[
B_{n} z = \sum_{k} (b_{nk} - b_{nk+1}) y_{k}, \ y \in m \text{ and } z \in m_{o}
\]

where \( y = \{y_{n}\} = \{ \sum_{v=0}^{n} z_{v} \} \).

Further if

\[
\lim \sum_{n} \sum_{k} |b_{nk} - b_{n,k+1}| = 0 \tag{3.3}
\]
then, since, for \( y \in m \) (that is, \( z \in m_0 \))

\[
|B_n z| \leq \|y\| \sum_k |b_{nk} - b_{n, k+1}|,
\]

it follows that, in the case (3.3) holds, \( \lim_{n} B_n z = 0 \) (\( z \in m_0 \)). Hence

in the case \( m_{B_0} \subset m_0 \) the requirement (3.2) is fulfilled. Now I am in a position to state the first theorem:

**Theorem 1:** Let \( B \) be a normal matrix such that condition (3.2) holds. Then for a row finite matrix \( A \)

\[
L(Ax) \leq \Omega_B(x) \quad (x \in m_B)
\]

(3.4)

if and only if \( AB^{-1} \) is almost positive and strongly regular.

**Remark:** By taking \( B = I \) (identity matrix) we obtain Theorem 3 of [3] for a row finite matrix.

For the proof of Theorem 1, I need to prove the following proposition which gives a theorem similar to the Knopp’s Core theorem in the case \( B = I \) and \( A \), a row finite matrix.

**Proposition 1:** Let \( B \) be a normal matrix. Then for a row finite matrix \( A \),

\[
\lim_{n} \sup A_n(x) \leq \lim_{n} \sup B_n(x) \quad \text{for all } x \in m_B
\]

(3.5)

if and only if \( AB^{-1} \) is regular and almost positive.

**Proof:** (Sufficiency) Since \( B \) is a normal matrix (see [5]), it is row finite and \( B^{-1} \) is also row finite. Let \( C = AB^{-1} \). Since \( CBx = (AB^{-1}) Bx = A(B^{-1}B) x = Ax \), it follows that

\[
L(Ax) = L(CBx)
\]

(3.6)

The associative property of infinite matrices \( A \), \( B^{-1} \) and \( B \) is justified for row finite matrices (see Cooke [2]). By the sufficiency part of Lemma 1,

\[
L(Cy) \leq L(y) \quad \text{for all } y \in m.
\]

Since for all \( x \in m_B \), \( Bx \in m \), we have from the above inequality that

\[
L(CBx) \leq L(Bx).
\]

As \( L(Ax) = L(CBx) \) by (3.6), we have proved the sufficiency.
Necessity: \(-L(\neg Bx) \leq -L(\neg Ax) \leq L(Ax) \leq L(Bx), (x \in m_B)\). Hence it follows that
\[L(Bx) = -L(\neg Bx) \Rightarrow L(Ax) = -L(\neg Ax),\]
that is,
\[\{x: Bx \in c\} \subset \{x: Ax \in c\}\]
and
\[\lim_{n} B_{nx} = \lim_{n} A_{nx}. \quad (3.7)\]

If \(y \in c\), then \(y \in m\). As \(B\) is normal there is an \(x \in s\) such that \(Bx = y\) or \(x = B^{-1}y\). Now by using (3.7) we have,
\[\lim y_n = \lim B_{nx} = \lim A_{nx} = \lim A_n(B^{-1}y) = (AB^{-1})_ny = \lim C_ny\]
Hence \(C = AB^{-1}\) is a regular matrix.

Now since \(C\) is regular, the requirement of lemma 2 is satisfied. Hence there exists \(y \in m\): \(\|y\| \leq 1\) and
\[L(Cy) = \lim sup_n \sum_k |c_{nk}|. \quad (3.8)\]
Now given \(y\) as above, define \(x\) by \(x = B^{-1}y\) so that \(\|Bx\| \leq 1\). Since \(L(Bx) \leq 1\), using (3.5) and (3.6) we get
\[L(Ax) = L(Cy) \leq 1. \quad (3.9)\]
Now it follows from (3.8) and (3.9) that
\[\lim sup_n \sum_k |c_{nk}| \leq 1. \quad (3.10)\]
But since
\[\lim sup_n \sum_k |c_{nk}| \geq \lim sup_n \sum_k c_{nk} = 1\]
it follows from (3.10) that
\[\lim sup_n \sum_k c_{nk} = 1.\]
Hence \(C\) is almost positive.

This completes the proof of the proposition.

Proof of the Theorem 1: (Sufficiency) Since \(C = AB^{-1}\) is almost positive and regular, it follows, by proposition 1, that
\[L(A(x+z)) \leq L(B(x+z)) (x \in m_B, z \in m_{B_0}).\]
Now taking the infimum with respect to \( z \in m_{B_0} \) in the above inequality, we have
\[
\Omega_A(x) \leq \Omega_B(x). \tag{3.11}
\]
Since \( L(Ax) \) is sublinear, it follows that
\[
\Omega_A(x) \geq \inf_{z \in m_{B_0}} |L(Ax) - L(-Az)|. \tag{3.12}
\]
But for \( z \in m_{B_0}, Az = CBz = Dy \)
where
\[
D = (d_{nk}) = (e_{nk} - e_{n;k+1}),
\]
\[
y = \{y_n\} = \{ \sum_{v=0}^{n} B_v(z) \} \in m. \tag{3.13}
\]
Since \( C \) is strongly regular and \( y \in m \), it follows that
\[
L(Az) = L(Dy) = 0.
\]
Hence it follows from (3.12) that
\[
\Omega_A(x) \geq \inf_{z \in m_{B_0}} L(Ax) = L(Ax). \tag{3.14}
\]
Now the sufficiency follows from (3.11) and (3.14).

*Necessity:* Suppose that (3.4) holds, since trivially \( \Omega_B(x) \leq L(Bx), \)
\( (x \in m_B) \) it follows from (3.4) that \( L(Ax) \leq L(Bx) \ (x \in m_B) \). Hence by proposition 1, \( C = AB^{-1} \) is almost positive and regular.

Since we know (see Devi [3], Theorem 1 (i))
\[
\Omega_B(x) = 0 (x \in m_{B_0}), \text{ it follows from (3.4) that}
\]
\[
L(Ax) \leq 0 \ (x \in m_{B_0});
\]
that is,
\[
L(CBx) \leq 0 \ (x \in m_{B_0});
\]
that is,
\[
L(Dy) \leq 0 \ (y \in m), \tag{3.15}
\]
where \( D \) and \( y \) are given by (3.13).

Now since the matrix \( D \) satisfies the conditions of Lemma 2 (as \( C \) is regular) there exists \( y_0 \in m: \|y_0\| \leq 1 \) and
\[ L(Dy_0) = \lim_{n, \sup} \sum_k |d_{nk}| \geq 0 \quad (3.16) \]

Now define \( x_0 \) by
\[ x_0 = B^{-1} (\sigma y_0 - y_0), \]
so that
\[ \sigma y_0 - y_0 = Bx_0. \]

Hence \( y_0 \in m \iff Bx_0 \in m_0 \iff x_0 \in m_{B0}. \)

Now taking \( y \) to be \( y_0 \) in (3.15) together with relation (3.16), we have
\[ \lim_{n} \sum_k |d_{nk}| = \lim_{n} \sum_k |c_{nk} - c_{nk+1}| = 0. \]

Hence \( C \) is strongly regular.

This completes the proof.

**Corollary 1:** Let the conditions of Theorem 1 hold. Then
\[ L(Ax) \leq \Omega_A(x) \leq \Omega_B(x) \leq L(Bx). \]

**Proof:** First inequality follows from (3.14), second inequality from (3.11) and the last one is trivial.

4. It is easy to see that
\[ \Omega_A(x) \leq L(Ax) \quad (x \in m) \quad (4.1) \]
but by Theorem 3 of Devi [3], we have
\[ L(Ax) \leq w(x) \quad (x \in m) \quad (4.2) \]
if and only if \( A \) is almost positive and strongly regular. Hence combining (4.1) and (4.2) we have

**Theorem 2:** Let \( A \) be almost positive and strongly regular. Then
\[ \Omega_A(x) \leq w(x) \quad (x \in m). \quad (4.3) \]

In other words,
\[ \{m, \Omega_A\} \prec \beta, \]
that is, \( \Omega_A \) generates Banach limits. This is justified as
\[ \beta = \{m, w\} \quad (\text{see Theorem 1'(ii) [3]}). \]

It is clear from (4.3) that
-w(-x) \leq -\Omega_A(-x) \leq \Omega_A(x) \leq w(x) \ (x \in m)

Since w(x) = -w(-x) implies that -\Omega_A(-x) = \Omega_A(x), it follows that
\{x \in m: w(x) = -w(-x)\} \subset \{x \in m: \Omega_A(x) = -\Omega_A(-x)\}.

that is,
\hat{c} \subset S_1,

if A is almost positive and strongly regular, where
\hat{c} = \{x \in m: x has unique Banach limit\}
= \{x \in m: w(x) = -w(-x)\},
= \left\{ x \in m: \frac{x_n + x_{n+1} + \ldots + x_{n+p}}{p + 1} \to \text{a limit as } p \to \infty, \text{ uniformly in } n \right\},

and
\begin{align*}
S_1 &= \{x \in m: \Omega_A(x) = -\Omega_A(-x)\}.
\end{align*}

\hat{c} is called the set of all almost convergent sequences (see Lorentz [4]).

In what follows, we want to examine if the set S_1 can have a simpler characterisation like the set \hat{c}.

Write
\begin{align*}
S_0 &= \{x \in m: \sum_k a_{nk} (x_k + z_k) \text{ converges uniformly in } z \in m_0\} \\
S_2 &= \{x \in m: \sum_k a_{nk} (x_k + z_k) \text{ converges for all } z \in m_0\}.
\end{align*}

We now prove

\textbf{Theorem 3:}

(i) \(S_0 \subset S_1\)

(ii) \(S_1 \subset S_2\) if \(\sum_k |a_{nk} - a_{nk+1}| \to 0\) as \(n \to \infty\).

\textbf{Proof:} Given \(x \in S_0\) and \(\varepsilon > 0\), there exists a positive integer \(n_0 = n_0(\varepsilon)\):
\begin{align*}
&\quad \quad s_1 - \varepsilon < \sum_k a_{nk} (x_k + z_k) < s_1 + \varepsilon \quad (4.4)
\end{align*}

for all \(z \in m_0\) and for all \(n \geq n_0\), where
\[ s_1 = \lim_{n} \sum_{k} a_{nk} (x_k + z_k), \]

and \( s_1 \) is independent of \( x \in m_0 \). Taking \( \lim \sup \) over \( n \) and then the infimum over \( z \) in (4.4) we obtain:

\[ s_1 - \varepsilon \leq - \Omega_A (-x) \leq \Omega_A (x) \leq s_1 + \varepsilon \]

(4.5)

Since \( \varepsilon \) is arbitrary, we get the first inclusion relation.

Next suppose that \( x \in S_1 \) and \( \Omega_A (x) = - \Omega_A (-x) = s_1 \). From \( \Omega_A (x) = s_1 \), we obtain, given \( \varepsilon > 0 \) there exists \( z' \in m_0 \) and \( n_1 \in \mathbb{N} \):

\[ A_n (x + z') = \sum_{k} a_{nk} (x_k + z'_k) < s_1 + \varepsilon, \]

(4.6)

for all \( n \geq n_1 \). Now for \( z \in m_0 \),

\[ A_n (x + z) = A_n (x + z') + A_n (z - z'). \]

(4.7)

Since \( \sum_{k} |a_{nk} - a_{n,k+1}| \to 0 \) as \( n \to \infty \), we obtain

\[ A_n (z - z') \to 0 \text{ as } n \to \infty, \]

that is, given \( \varepsilon > 0 \), there exists \( n_2 \in \mathbb{N} \):

\[ A_n (z - z') < \varepsilon \text{ (n \geq n_2)}. \]

(4.8)

Now from (4.6), (4.7) and (4.8) we have

\[ A_n (x + z) < s_1 + 2 \varepsilon \text{ for all } n > n_3 = \max (n_1, n_2). \]

Similarly we have

\[ A_n (x + z) > s_1 - 2 \varepsilon \text{ for all } n \geq n_4 \in \mathbb{N}. \]

so that

\[ |A_n (x + z) - s_1| < 2 \varepsilon \text{ for all } n \geq n_5 = \max (n_3, n_4). \]

Hence \( x \in S_2 \) and this proves the second inclusion relation.

5. The set \( S_0 \) defined in Section 4 is usually empty. In fact it is non-empty only if the matrix \( A \) has a finite number of non-zero rows. In view of this it is evident that the inclusion (i) of Theorem 3 is proper because when \( A \) is almost positive and strongly regular then \( \hat{c} \subset S_1 \) (see below Theorem 2).

Now the natural question arises as to what sublinear functional \( \Upsilon \) will generate the set \( S_0 \) in the sense that
$S_0 = \{ x \in m : \Psi^*(x) = -\Psi^*(-x) \}.$

Towards this end, we define $\Psi^*_A : m \to \mathbb{R}$ by

$$\Psi^*_A(x) = \lim sup_n \sup_{z \in V} \sum_k a_{nk} (x_k + z_k)$$

where $V$ is a subspace of $m$.

Since

$$\Psi^*_A(x) \geq \lim sup_n \sum_k a_{nk} x_k$$

and if $x \in m$, $\|A\| < \infty$ then $\Psi^*_A$ is bounded from below. $\Psi^*_A$ is also bounded from above if $V$ is a bounded subspace. In this case $\Psi^*_A$ is well-defined. Now we have the following

**Theorem 4:** Let $V$ be a bounded subspace of $m$ and let $\|A\| < \infty$.

Write

$$\hat{S}_0 = \{ x \in m : \sum_k a_{nk} (x_k + z_k) \to a \text{ limit as } n \to \infty \text{ uniformly in } z \in V \}.$$ 

Then

$$\hat{S}_0 = \{ x \in m : \Psi^*_A(x) = -\Psi^*_A(-x) \}.$$

**Proof:** Suppose that $\sum_k a_{nk} (x_k + z_k) \to z$ uniformly in $z \in V$.

Then given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$:

$$\alpha - \varepsilon < \sum_k a_{nk} (x_k + z_k) < \alpha + \varepsilon$$

for all $z \in V$, $n \geq n_0$.  

Now taking $\sup$ with respect to $z$ and then $\lim sup$ with respect to $n$, we have

$$\alpha - \varepsilon \leq \Psi^*_A(-x) \leq \Psi^*_A(x) \leq \alpha + \varepsilon.$$ 

Since $\varepsilon$ is arbitrary, we obtain

$$\Psi^*_A(x) = -\Psi^*_A(-x) = \alpha.$$ 

Conversely suppose that $\Psi^*_A(x) = z = -\Psi^*_A(-x)$. Then we shall have

$$\alpha - \varepsilon < \sum_k a_{nk} (x_k + z_k) < \alpha + \varepsilon$$

for all $z \in V$, $n \geq n_0$, from which follows that

$$\sum_k a_{nk} (x_k + z_k) \to \alpha \text{ uniformly in } z \in V.$$ 

This completes the proof.
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