On Circulant Matrices

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SUMMARY

In this article rows, we introduce a more generalized notion of circulant matrix, namely, $q$-rows $l$-circulant matrices. Suppose the matrix is $n \times n$ and $q$ is a divisor of $n$ so that the rows of the matrix can be partitioned into blocks of $q$-rows each. Let each row block be obtained from the preceding one by shifting all its entries $l$ places to the right. In generalizing the study to those $q$-row $l$-circulant matrices, we have obtained as special cases the results on $1$-row $l$-circulant and $1$-row $1$-circulant matrices.

1. INTRODUCTION

Circulant matrices have a long past [1]. B. Friedman has studied the eigenvalues and canonical forms of composite matrices [2]. C. M. Ablow and J. L. Brenner have studied the canonical forms of $1$-row and $g$-circulant matrices [3].

In this paper we shall give new proofs of some theorems which have already been proved. We shall also give a new concept about circulant matrices which are not composite and continuant matrices [1, 2]. By generalizing the notion of circulant matrix (see §2, Definitions 2, 3) we find it possible to locate all the roots and describe all of the vectors of the type of matrix that occurs in the Hurwitz-Routh theory; see the illustration on page 24 of this article.

This paper discusses different kinds of circulant matrices and some of their properties.

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2. Definition 1. Let $P_n$ be the following $nxn$ matrix.

\[
\begin{bmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix}
\]

Lemma 1. $P_n^n = I$

Definition 2. Let $R_q$ be a $qxn$ matrix, where $q < n$.

\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{q1} & a_{q2} & \cdots & a_{qn}
\end{bmatrix}
\]

Definition 3. Let $R_q$ be a matrix of order $qxn$, and $g, r, q$ be positive integers such that $1 \leq g \leq n$, $n = qr$.

$A$ is the $nxn$ matrix

\[
A = \begin{bmatrix}
R_q P_n & R_q P_n^q & \cdots & R_q P_n^{qg} \\
R_q P_n & R_q P_n^{2qg} & \cdots & R_q P_n^{(r-1)qg} \\
\vdots & \vdots & \ddots & \vdots \\
R_q P_n & R_q P_n^{(r-1)qg} & \cdots & R_q P_n^{(r-1)qg}
\end{bmatrix}
\]

$R_q P_n^{(v-1)qg}$ is called the $v$-th row block of $A$.

In the general case, if the $v$-th row block of $A$ is $R_q P_n^{(v-1)qg}$, we shall call $A$ a $q$-rows $l$- circulant. Thus we have the following definitions.

(i) If the $v$-th row block of $A$ is $R_q P_n^{(v-1)qg}$, $A$ is called a $q$-rows $qg$-circulant matrix. (Called a g-cycle matrix by Friedman [2].)

(ii) If $g = 1$ and the $v$-th row block of $A$ is $R_q P_n^{(v-1)q}$, $A$ is called a $q$-rows $q$-circulant matrix (continuant matrix)

(iii) If $q = 1$ and the $v$-th row $A$ is $R_{1} P_n^{(v-1)g}$, $A$ is called a $1$- row $g$-circulant matrix (called a g-circulant matrix by J. L. Brenner [3].)
Theorem 1. A necessary and sufficient condition for a matrix $A$ to be $q$-rows $qg$-circulant is

\[(1) \quad P_n^q A = A P_n^{qg} \]

Proof. The condition is necessary. Let $A$ be a $q$-rows $qg$-circulant matrix. If any $nxn$ matrix $B$, where $n$ is $qr$, is multiplied on the left by $P_n^q$ then the second row block of $B$ is transformed into its first, the third block into the second, the $r$-th block into the $r$-1 st block and the first block into the $r$-th block. $A$ being $q$-rows $qg$-circulant we have by definition 3

\[
P_n^q A = \begin{bmatrix} R_q P_n^{qg} \\ R_q P_n^{2qg} \\ \vdots \\ R_q P_n^{(r-1)qg} \\ R_q \\ \end{bmatrix}, AP_n^{qg} = \begin{bmatrix} R_q P_n^{qg} \\ R_q P_n^{2qg} \\ \vdots \\ R_q P_n^{(r-1)qg} \\ R_q \\ \end{bmatrix}
\]

By lemma 1 we have $P_n^{qg} = P_n^{ng} = (P_n^n)^g = I$. Thus every $q$-row $qg$-circulant matrix satisfies the relation (1).

The condition is sufficient. We shall show that every matrix satisfying relation (1) is a $q$-rows $qg$-circulant matrix.

Let the row blocks of the matrix $A$ be $U_1, U_2, ..., U_r$. By (1) we have

\[U_{i+1} = U_i P_n^{qg}, \ i = 1, ..., r,\]

which shows that

\[U_1 = U_1, U_2 = U_1 P_n^{qg}, U_3 = U_1 P_n^{2qg}, ..., U_r = U_1 P_n^{(r-1)qg}; \] or

\[A = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_r \\ \end{bmatrix} = \begin{bmatrix} U_1 P_n^{qg} \\ U_1 P_n^{2qg} \\ \vdots \\ U_1 P_n^{(r-1)qg} \\ \end{bmatrix}\]

By definition 3, $A$ is a $q$-rows $qg$-circulant matrix.
Lemma 2. If $A$ is a $q$-rows $qg$-circulant matrix, then
\[ p_a^{-qg} A = A p_a^{-kqg} \]
where $k$ is a positive integer.

It can be proved by induction.

Theorem 2. The product of a $q$-rows $qg$-circulant matrix and a $q$-rows $qh$-circulant matrix is a matrix which is $q$-rows $qgh$-circulant.

Proof. We use theorem 1 and lemma 2. The special case for $q = 1$ is due to Ablow and Brenner [3]. Let $A$ be a $q$-rows $qg$-circulant matrix and let $B$ be $q$-rows $qh$-circulant. Considering theorem 1 and lemma 2 we have
\[ p_a^{-q} A = A p_a^{-qg}, \quad p_a^{-q} B = B p_a^{-qh}, \text{ and} \]
\[ p_a^{-q} AB = A p_a^{-qg} B = AB p_a^{-qgh}, \text{ or} \]
\[ p_a^{-q} AB = AB p_a^{-qgh} \]

Corollary. If $A$ and $B$ are $q$-rows $q$-circulant matrices then

(i) $p_a^{-q} A = A p_a^{-q}$

(ii) $p_a^{-q} AB = AB p_a^{-q}$

(iii) $p_a^{-q} A^k = A^k p_a^{-q}$, where $k$ is a positive integer.

Theorem 3. The inverses of $q$-rows $q$-circulant matrices are also $q$-rows $q$-circulant matrices.

Proof. Let $A$ be a non-singular $q$-rows $q$-circulant matrix of order $nxn$. Multiplying both sides of the relation $AA^{-1} = I$ by $p_a^{-q}$ form the left, we have
\[ p_a^{-q} AA^{-1} = p_a^{-q}, \quad A p_a^{-q} A^{-1} = p_a^{-q} \]
\[ p_a^{-q} A^{-1} = A^{-1} p_a^{-q} \]

Corollary. The adjoints of $q$-rows $q$-circulant matrices are $q$-rows $q$-circulant matrices.

Theorem 4. If $A$ is a $q$-rows $q$-circulant matrix, then
\[ p_a^{-q} f(A) = f(A) p_a^{-q} \]
where $f(x)$ is a polynomial in the scalar variable $x$. 
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Proof. It can be proved by the corollary of theorem 2.

Theorem 5. (The commutative property of the multiplication of the 1-row 1-circulant matrices). If A and B are 1-row 1-circulant matrices of order nxn, then we have

$$AB = BA$$

Proof. This is well-known [1], [4, p. 95]. Let A and B be given as follows.

$$A = \begin{bmatrix}
R \\
RP_a \\
\vdots \\
\vdots \\
RP_a^{n-1}
\end{bmatrix} \quad B = \begin{bmatrix}
R_i \\
R_i P_a \\
\vdots \\
\vdots \\
R_i P_a^{n-1}
\end{bmatrix}$$

where $$R = [a_{11} a_{12} \ldots a_{1n}]$$, $$R_i = [b_{11} b_{12} \ldots b_{1n}]$$.

By $$P_a A = A P_a$$ and $$P_a^k A = A P_a^k$$ for-th rows of AB and BA are $$RP_a^{n-1}$$ B = $$RBP_a^{n-1}$$ and $$R_i P_a^{n-1} A = R_i A P_a^{n-1}$$. Now we show $$RB$$ and $$R_i A$$ are the same.

$$RB = \begin{bmatrix}
R_i \\
R_i P_a \\
\vdots \\
\vdots \\
R_i P_a^{n-1}
\end{bmatrix} = [a_{11} R_i + a_{12} R_i P_a + \ldots + a_{1n} R_i P_a^{n-1}]$$ or

$$RB = R_i [a_{11} I + a_{12} P_a + \ldots + a_{1n} P_a^{n-1}] = R_i A$$

3. In order to obtain the eigenvalues and canonical forms of a q-rows i-circulant matrix, we shall be concerned with certain subsets of the set of residue classes modulo r defined as follows.

Definition 4. Let $$(g, r) = 1$$

$$g^k h_i \equiv h_{i+1} \pmod{r}$$

where $$k$$ is some positive integer.

If $$h_i$$ is any residue modulo r, then

$$h_i, h_i g, h_i g^2, \ldots, h_i g^{r-1} \pmod{r}$$
is a subset of the residues, where $t$ is the least positive integer such that $g^t \equiv 1 \pmod{r}$. Note that $t$ divides $\varphi(r)$. This kind of subset modulo $r$ is called by Friedman a minimal invariant set under multiplication by $g \pmod{r}$. This form of the definition is given by J. L. Brenner [3,5].

We note that if $(g, r) = 1$, then Euler's theorem shows that $t$ exists and satisfies the conditions $g^t = 1 \pmod{r}$, $t \mid \varphi(r)$. If $(g, r) > 1$, then no such integer $t$ exists [8, p. 50].

As an illustration of this definition, consider the case where $r = 21$, $g = 4$. Then $t = 3$ and the minimal invariant sets are the following:

$[0], [1, 4, 16], [2, 8, 11], [3, 12, 6], [5, 20, 17], [7, 9, 15, 18], [10, 19, 13], [14].$

We define the direct product of matrices [6]. Suppose that $A$ is an nx$n$ and $B$ an mxm matrix. Then the direct product $A \otimes B$ is the nmxnm matrix defined by

$$A \otimes B = \begin{bmatrix} b_{11}A & b_{12}A & \cdots & b_{1m}A \\ & & & \\ b_{m1}A & b_{m2}A & \cdots & b_{mm}A \end{bmatrix}$$

By the definition, we have

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = A_1A_2 \otimes B_1B_2$$

Lemma 3. Let $n = qr$; $(q, r) = 1$ and let $s$ be the $n$-th root of unity $s = \exp(2\pi i/n)$, and $X_n(s^h)$ be the column vector $(1, s^h, s^{2h}, \ldots, s^{(n-1)h})^T$. Then

(i) $P_nX_n(s^h) = s^hX_n(s^h); \ h = 1, \ldots, n.$

(ii) $P^{kh}_nX_n(s^h) = s^{kh}X_n(s^h), k$ is any positive integer.

(iii) The $r$-th roots of unity are

$s^{nh}; \ h = 1, \ldots, r$

(iv) $X_q(s^h) \otimes X_r(s^{nh}) = X_n(s^h)$

(v) $I_q \otimes P_r = P_n^q$
(vi) \( I_q \otimes P_x^k = P_a^{kq} \), \( k \) is any positive integer.

(vii) \( R_q (X_q \otimes X_r (s^{qh}) ) = M(s^{qh})X_q \) where \( X_q \) is an arbitrary column vector with \( q \) components and \( M(s^{qh}) \) is a matrix of order \( qxq \) defined by

\[
M(s^{qh}) = \sum_{v=1}^{r} A_v s^{(v-1)qh}, \text{ where}
\]

\[
A_v = \begin{bmatrix}
    a_1, (v-1)q+1 & a_1, (v-1)q+2 & \cdots & a_1, (v-1)q+q \\
    a_2, (v-1)q+1 & a_2, (v-1)q+2 & \cdots & a_2, (v-1)q+q \\
    \vdots & \vdots & \ddots & \vdots \\
    a_q, (v-1)q+1 & a_q, (v-1)q+2 & \cdots & a_q, (v-1)q+q 
\end{bmatrix}
\]

Theorem 6. Let \( A \) be \( q \)-rows \( qg \)-circulant matrix of order \( nxn \). Let \( h_1, h_g, \ldots, h_g \) be the distinct minimal invariant subsets of \( g(\text{mod} \: r) \); \( (g, r) = 1 \). Then the matrix \( A \) has the following representation,

\[
W_1 \oplus W_2 \oplus \cdots \oplus W_j,
\]

where \( B \oplus C \) denotes

\[
\begin{bmatrix}
    B & O \\
    O & C
\end{bmatrix}
\]

and

\[
W_i = \begin{bmatrix}
    0 & 0 & \cdots & 0 & M(s^{qh_1}) & 0 & \cdots & 0 \\
    0 & M(s^{qh_1}) & \cdots & 0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & M(s^{qh_{j-1}}) & 0 & \cdots & 0
\end{bmatrix}
\]

Proof. Let \( X_q \) be an arbitrary column vector with \( q \) components. We multiply both sides of the following relation on the right by \( X_q \otimes X_r (s^{qh}) \)

\[
A = \begin{bmatrix}
    R_q \\
    R_q P_a^{qg} \\
    \vdots \\
    R_q P_a^{(r-1)qg}
\end{bmatrix}
\]

Using lemma 3 and the direct product of matrices, we reduce the \( v \)-th row block of the product to the following form:
\[ R_q P_n (v^{-1})g \ (X_q \otimes X_r (s^{gb})) = R_q (I_q \otimes P_r (v^{-1})g) \ (X_q \otimes X_r s^{gb}) \] or
\[ R_q P_n (v^{-1})g \ (X_q \otimes X_r (s^{gb})) = R_q (X_q \otimes X_r (s^{gb})) s^{(v^{-1})g} h. \]

Therefore we have
\[ A(X_q \otimes X_r (s^{gb})) = (M(s^{gh}) X_q) \otimes X_q (s^{gb}) \]
\[ A(X_q \otimes X_r (s^{gb})) = (M(s^{gh}) X_q) \otimes X_r (s^{gb}) \]
\[ \ldots \]
\[ A(X_q \otimes X_r (s^{gb})) = (M(s^{gh}) X_q) \otimes X_r (s^{gb}) \]

or
\[ W_i = \begin{bmatrix}
0 & 0 & \ldots & 0 & M(s^{tb-1}h) \\
M(s^{gb}) & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & M(s^{tb-2}h) & 0 \\
\end{bmatrix}, i = 1, \ldots, j \]

where \( W_i \) is a broken diagonal matrix of order \( t \times t \). This result is due to Friedman [2].

Theorem 7. If \( A \) is a \( g \)-rows \( qg \)-circulant matrix, then
\[ AX_n (s^b) = (M(s^{ab}) X_q (s^b)) \otimes X_r (s^{gb}) \]

Prof. By lemma 3 (i), (ii) and (iv), and (vii)
\[ AX_n (s^b) = R_q X_n (s^b) \otimes X_r (s^{gb}) \]
\[ AX_n (s^b) = M(s^{gb}) X_q (s^b) \otimes X_r (s^{gb}) \]

As a special case for \( q = 1, g = 1 \), we have
\[ AX_n (s^b) = (a_{11}^1 + a_{12}^1 s^b + \ldots + a_{1n}^1 s^{(n-1)b}) X_n (s^b) \]

This is Abelow and Brenner's Theorem [3].

4. In the present section we shall investigate the \( q \)-rows \( g \)-circulant matrices. The type of matrices that occur in the Hurwitz-Routh theory [7; v. 2, Chapter XV].

Theorem 8. If \( A \) is a \( q \)-rows \( g \)-circulant matrix, then
\[ AX_n (s^b) = M (s^{ab}) X_q (s^b) \otimes X_r (s^{gb}) \]

Proof. We multiply both sides of the following relation on the right by \( X_n (s^b) \).
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\[
A = \begin{bmatrix}
R_q \\
R_q P_n^g \\
. \\
. \\
R_q P_n^{(r-1)g}
\end{bmatrix}
\]

Using lemma 3 and the direct product of matrices, we reduce the \(v\)-th row of the product to the following form:

\[
R_q P_n^{(v-1)g} X_n(s^b) = R_q X_n(s^b) s^{(v-1)g} = M(s^{g}) X_q(s^b) s^{(v-1)g}
\]

Therefore we have

\[
AX_n(s^b) = M(s^{g}) X_q(s^b) \otimes X_r(s^g)
\]

We shall discuss an application of theorem 8 and shall investigate the Hurwitz matrices [7; v. 2, p. 190].

A Hurwitz matrix \(H\) may be defined as follows. Let.

\[
R_2 = \begin{bmatrix}
b_0 & b_1 & b_2 & \cdots & b_{n-1} \\
a_0 & a_1 & a_2 & \cdots & a_{n-1}
\end{bmatrix}
\]

be a matrix of order \(2 \times n\), where \(n = 2r\) and

\[
a_k = 0 \text{ for } k > r \\
b_k = 0 \text{ for } k > r-1
\]

We then define \(H\) by

\[
H = \begin{bmatrix}
R_q \\
R_q P_n \\
. \\
. \\
R_2 P_n^{r-1}
\end{bmatrix}
\]

This is a 2-rows 1- circulant matrix.

Theorem 9. If \(H\) is a Hurwitz matrix of order \(n \times n\) where \(n = 2r\). Then.

\[
HX_n(s^b) = M(s^{g}) X_2(s^b) \otimes X_r(s^b)
\]

Proof. This is a special case of theorem 8 for \(q = 2, g = 1\).

We have

\[
HX_n(s^b) = M(s^{g}) X_2(s^b) \otimes X_r(s^b)
\]
As an illustration of this theorem consider the Hurwitz matrix of order 6x6, \( q = 2, r = \cdot 3 \).

\[
\begin{bmatrix}
- b_0 & b_1 & b_2 & 0 & 0 & 0 \\
 a_0 & a_1 & a_2 & a_3 & 0 & 0 \\
 0 & b_0 & b_1 & b_2 & 0 & 0 \\
 0 & a_0 & a_1 & a_2 & a_3 & 0 \\
 0 & 0 & b_0 & b_1 & b_2 & a_3 \\
 0 & 0 & a_0 & a_1 & a_2 & a_3
\end{bmatrix}

\begin{bmatrix}
1 \\
s^h \\
s^{2h} \\
s^{3h} \\
s^{4h} \\
s^{5h}
\end{bmatrix}

= \begin{bmatrix}
- b_0 + b_2 s^{2h} & b_1 \\
 a_0 + a_2 s^{3h} & a_1 + a_3 s^{2h}
\end{bmatrix}

\begin{bmatrix}
1 \\
s^h \\
s^{2h}
\end{bmatrix}

\otimes

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REFERENCES


ÖZET

Bu çalışmada, sirkülant matrisler daha genel anlamda \( q \)-satuhr \( l \)-sirkülant matrisler olarak ifade edilmiştir. \( q, n \) nin bir bölüne olmak üzere \( n \times n \) mertebeden bir matris \( q \) satırlı bloklara bölünmüş olduğunu kabul edelim. Her satır blok, bir önceki satır bloğun sağdan \( l \) tane elemanını sol tarafa yer değiştirmesile elde edilmiş olun,

Bu genelleştirme ile \( q \)-satuhr \( l \)-sirkülant ve \( 1 \)-satuhr \( 1 \)-sirkülant matrislere ait teoremler. \( q \)-satuhr \( l \)-sirkülant matrislere ait teoremlerin özel halleri olduğu görülür.
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