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On Cesaro Sums of Divergent Series

by

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On Cesaro Sums of Divergent Series

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SUMMARY

Let $\sum_{k=1}^{\infty} a_k$ be an infinite series of real, non-negative numbers and let

$$(\varepsilon) = \{\varepsilon_k\}, (k=1,2,\ldots, \varepsilon_k = \pm 1)$$

be any sequence of signs.

For a given sequence $(\varepsilon)$, we denote the $n$-th partial sum of the series $\sum \varepsilon_k a_k$ by

$$s_n (\varepsilon) = \sum_{k=1}^{n} \varepsilon_k a_k$$

and the $n$-th partial $C_1$-sum of the series by

$$\sigma_n (\varepsilon) = \frac{1}{n} \sum_{v=1}^{n} s_v (\varepsilon).$$

If $\sigma_n (\varepsilon)$ converges then we call

$$\sigma (\varepsilon) = \lim_{n \to \infty} \sigma_n (\varepsilon)$$

a $C_1$-attainable point of $\Sigma a_k$ and denote the set of all $C_1$-attainable points of $\Sigma a_k$ by $SC (a_k)$.

In this paper we are going to investigate the $C_1$-attainable set $SC (a_k)$ of a divergent series $\Sigma a_k$ and give some theorems on that $SC (a_k) = \mathbb{R}$ and $SC (a_k) = \emptyset$, where $\mathbb{R}$ is the set of real numbers and $\emptyset$ is the empty set.
1. Introduction

It is known that, if a numerical series is conditionally convergent, then it is possible to sum this series to any value by rearranging its terms, [4], [5].

A similar problem has been investigated for divergent series and some interesting results have been obtained by Bagemihl-Erdős, [3]. Also, Erdős-Hanani got some results for the $C_i$-attainable set of a divergent series $\sum a_k$, [1].

In this note we are going to deal with the same type of problems.

2. Notations.

Let $\sum_{k=1}^{\infty} a_k$ be an infinite series of real, non-negative numbers and let

$$(2.1) \quad (\varepsilon) = \{\varepsilon_k\}, \quad (k=1,2,\ldots, \varepsilon_k = \pm 1)$$

be any sequence of signs.

For a given sequence $(\varepsilon)$, we denote the $n$-th partial sum of the series $\sum \varepsilon_k a_k$ by

$$s_n(\varepsilon) = \sum_{k=1}^{n} \varepsilon_k a_k$$

and the $n$-th partial $C_i$-sum of the series by

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If $\sigma_n(\varepsilon)$ converges then we call

$$\sigma(\varepsilon) = \lim_{n \to \infty} \sigma_n(\varepsilon)$$

a $C_i$-attainable point of $\Sigma a_k$ and denote the set of all $C_i$-attainable points of $\Sigma a_k$ by $SC(a_k)$.

$R$ will denote the set of real numbers and $\varnothing$ will denote the empty set.
3. Theorems For SC \((a_k) = R\).

Let us start giving a theorem which is an immediate consequence of Theorem 1 of Erdős – Hanani [1] and Theorem 3 of Yurtsever, [2].

**Theorem 3.1.** Let \(\Sigma a_k\) be a series of nonnegative terms having a subseries \(\Sigma a_{n_k}\) such that
\[
\Sigma a_{n_k} = \infty, \ a_{n_k} \to 0.
\]
If \((a_k)\) is monotone and bounded then SC \((a_k) = R\). (2)

**Theorem 3.2.** Let \(\Sigma a_k = \infty\) be a series of non-negative terms having a subseries \(\Sigma a_{n_k}\) such that
\[
\Sigma a_{n_k} = \infty, \ a_{n_k} \to 0.
\]
If, for a definite sequence \((\varepsilon)\),
\[
a) \lim_{k \to \infty} \frac{1}{k+1} \Sigma \varepsilon_{n+1} \varepsilon_{n+1} \text{ exists,}
\]
and

(2) During my stay in University of Lancaster in 1969–71, Prof. I. J. Maddox suggested me that Theorem 3.1. can be improved to the following

**Theorem 3.1’.** Let \(\Sigma a_k\) be a series of non-negative terms having a subseries \(\Sigma a_{n_k}\) such that
\[
\Sigma a_{n_k} = \infty, \ a_{n_k} \to 0.
\]
If \(\Sigma |\Delta a_k| = \Sigma |a_k - a_{k+1}| < \infty\), then SC \((a_k) = R\).

**Proof.** Take \(\varepsilon_k = (-1)^k\). Then \(\Sigma \varepsilon_k a_k\) is convergent (and so \((C,1)\) summable), for
\[
\frac{n}{k=0} \Sigma \varepsilon_k a_k = a_n \left( \frac{n}{k=0} \Sigma \varepsilon_k \right) + \frac{n-1}{k=0} \left( \frac{k}{\mu=0} \Sigma \varepsilon_{\mu} \right) \Delta a_k.
\]
Now \(\Sigma |\Delta a_k| < \infty\) implies that \(a_n\) tends to a limit, \(l\), say, as \(n \to \infty\). But \(a_{n_k} \to 0\) implies that \(l = 0\), i.e., \(a_n \to 0\).

Hence
\[
\frac{n}{k=0} \Sigma \varepsilon_k a_k = \frac{n}{k=0} \Sigma \varepsilon_k + \frac{n-1}{k=0} \left( \frac{k}{\mu=0} \Sigma \varepsilon_{\mu} \right) \Delta a_k
\]
\[
= o(1) \ 0 \ (1) + \frac{n-1}{k=0} \left( 0 \ (1) \right) \Delta a_k.
\]
So the result is immediate.
b) the series $\sum s_v \Delta \varepsilon_v$ is $C_1$ – summable, where

$$s_v = \sum_{\mu=0}^{v} a_{\mu}, \text{ and } \Delta \varepsilon_v = \varepsilon_v - \varepsilon_{v+1}, \text{ then } SC (a_k) = R.$$  

**Proof.** Take the series $\Sigma a_k = \infty$ and apply the sequence (e). According to the Abel partial summation formula, we have

$$\sum_{k=0}^{n} \varepsilon_k a_k = \sum_{k=0}^{n} s_k \Delta \varepsilon_k + s_n \varepsilon_{n+1},$$

where $s_n = \sum_{\mu=0}^{n} a_{\mu}$, $s_{-1} = 0$ and $\Delta \varepsilon_k = \varepsilon_k - \varepsilon_{k+1}$.

If we put

$$S_j = \sum_{k=0}^{j} \varepsilon_k a_k = \sum_{k=0}^{j} s_k \Delta \varepsilon_k + s_j \varepsilon_{j+1}, \text{ (j=0,1,2,...)},$$

we easily get

$$\lim_{j \to \infty} \frac{S_0 + S_1 + \ldots + S_j}{j + 1}$$

$$= \lim_{j \to \infty} \frac{1}{j+1} \sum_{k=0}^{j} s_k \varepsilon_{k+1} +$$

$$s \Delta \varepsilon_0 + \sum_{k=0}^{j} s_k \Delta \varepsilon_k + \ldots + \sum_{k=0}^{j} s_k \Delta \varepsilon_k$$

$$\lim_{j \to \infty} \frac{1}{j + 1}$$

Since the left-hand side of (3.2) is the $C_1$–sum of the series $\Sigma \varepsilon_k a_k$, by Theorem 1 of Erdős-Hanani, [1], the result is straightforward.

**Theorem 3.3.** Let $\Sigma a_k$ be a series of non-negative terms. If $\Sigma a_k = \infty$ and monotonously $a_k \to 0$, then $SC (a_k) = R$.

**Proof.** Let us write the equality (3.1) in the form of

$$\sum_{k=0}^{n} \varepsilon_k a_k = \sum_{k=0}^{n} s_k \Delta a_k + s_n a_{n+1},$$

$$\sum_{k=0}^{n} \varepsilon_k a_k = \sum_{k=0}^{n} s_k \Delta a_k + s_n a_{n+1},$$
where \( s_n = \sum_{\mu=0}^{n} \varepsilon_{\mu}, \ s_{-1} = 0, \) and \( \Delta a_k = a_k - a_{k+1} \).

Put

\[ S_j = \sum_{k=0}^{j} \varepsilon_k a_k. \]

Now, if

\[ \text{a') } \lim_{j \to \infty} \frac{1}{j+1} \sum_{k=0}^{j} s_k a_{k+1} \text{ exists,} \]

and

\[ \text{b') } \text{the series } \sum s_k \Delta a_k \text{ is } C_1 - \text{summable,} \]

then

\[ (3.4) \lim_{j \to \infty} \frac{S_0 + S_1 + \ldots + S_j}{j+1} \]

exists.

So, we must show, under the given hypothesis, that conditions a') and b') are satisfied.

Choose \( \{\varepsilon_k\} = (-1)^k, \ (k = 0, 1, 2, \ldots) \). Then the partial sums \( s_k \)'s are bounded, and since \( a_k \to 0 \) monotonously, the series \( \sum s_k \Delta a_k \) is convergent. (One can easily see that it is absolutely convergent, in fact.) So, condition b') is satisfied. Namely SC \( (a_k) \neq \emptyset \). Condition a') is also satisfied because of the Cauchy's Theorem. The limit exists and equal to zero, (Arithmetic Means), [4], [5].

Therefore, according to Theorem 1 of Erdös-Hanani, [1], SC \( (a_k) = R \).

4. A Problem of Erdös - Hanani

In this section, we are going to consider a problem due to Erdös-Hanani, (Problem 1, [1]), and show that the best possible result is \( C = 1 \).
Theorem 4.1. Let $\Sigma a_k$ be a series of nonnegative terms satisfying $\Sigma a_k = \infty$. If there exists an $\eta_0$ with the property that to each $\eta$ in $0 < \eta \leq \eta_0$ there corresponds an

(4.1) $n_\eta = n_\eta(\eta)$

such that for every $n > n_\eta$,

(4.2) $\sum_{i=1}^{\lceil \eta n / a_n \rceil} a_{n+i} > a_n + \eta$

then $SC(a_k) = R$.

Proof. Let $\sigma$ be any real number. Then, we are going to construct a sequence (2.1) such that

$$\lim_{n \to \infty} \sigma_n(\varepsilon) = \sigma.$$  

According to (4.1), for every $\eta_i = 2^{-i}, (i = i_0, i_0 + 1, \ldots)$ there exists a number

(4.3) $n_i = n_i(2^{-i})$

such that for every $n > n_i$, (4.2) is satisfied, with $\eta = 2^{-i}$.

Now, choose $\varepsilon_j$ arbitrarily for $j = 1, 2, \ldots, n_{i_0-1}$. Then, let us put $n_i = j$ and suppose that

$$\sigma_{j-1}(\varepsilon) = \frac{s_1(\varepsilon) + \ldots + s_{j-1}(\varepsilon)}{j - 1} \leq \sigma.$$

If $s_{j-1}(\varepsilon) \leq \sigma + 2^{-j}$ we take $\varepsilon_j = +1$ to make $\sigma_j(\varepsilon)$ bigger than $\sigma$. But, if $s_{j-1}(\varepsilon) > \sigma + 2^{-j}$, then we choose $\varepsilon_j$ so as to make $s_j(\varepsilon)$ as small as possible but not less than $\sigma + 2^{-j}$. Continuing this way, suppose that the final partial sum we reached is $s_{k_1}(\varepsilon)$ and let

$$\sigma_{k_1}(\varepsilon) = \frac{s_1(\varepsilon) + s_2(\varepsilon) + \ldots + s_{k_1}(\varepsilon)}{k_1}.$$

Now the means must start decreasing and be $\leq \sigma$. Therefore the partial sums must decrease. Then if

$s_{k_1}(\varepsilon) \geq \sigma - 2^{-i}$, we put $\varepsilon_{k_1+1} = -1$;

but, if $s_{k_1}(\varepsilon) < \sigma - 2^{-i}$, we choose $\varepsilon_{k_1} + 1$ so as to make the left hand side as large as possible but not greater than $\sigma - 2^{-i}$. 
Accordingly, we get
\[ \sigma_{j_2}(\varepsilon) = \frac{s_1(\varepsilon) + \ldots + s_{k_1}(\varepsilon) + \ldots + s_{j_2}(\varepsilon)}{j_2} \leq \sigma. \]

Then, it follows that the sequence \( (\sigma_{j_2}(\varepsilon) ) \) attains alternately minima \( \sigma_{j_h}(\varepsilon) \), \( (h = 1, 2, \ldots) \) and maxima \( \sigma_{k_h}(\varepsilon) \), \( (h=1,2,\ldots) \), with \( j_1 < k_1 < j_2 < k_2 < \ldots \) such that
\[ \sigma_{j_h}(\varepsilon) \leq \sigma \text{ and } \sigma_{k_h}(\varepsilon) > \sigma. \]

Therefore the sequence \( (\sigma_{j_2}(\varepsilon) ) \) is monotonically increasing for \( j_h \leq u \leq k_h \) and monotonically decreasing for \( k_h \leq u \leq j_{h+1} \).

To prove the theorem, it is enough to show that the difference between \( \sigma \) and maxima \( \sigma_{k_h}(\varepsilon) \) (or, \( \sigma \) and minima \( \sigma_{j_h}(\varepsilon) \) ) tends to zero as \( n \to \infty \). So we must show the existence of a number \( j_0 \) such that for every \( k_h > j_0 \)
\[ (4.4) \quad 0 < \sigma_{k_h}(\varepsilon) - \sigma < \eta \]
holds.

Let \( i \) be an integer such that
\[ (4.5) \quad 2^{-i} < \eta/6 \]
and let \( n_i \) be the corresponding index fixed by \( (4.1) \):
\[ n_i = n_i \left(2^{-i}\right). \]

Further, let \( h \) be an integer such that \( k_{h-1} > n_i \) and \( m \) the greatest index providing \( j_h < m \leq k_h \) such that \( \varepsilon_m = 1 \).

According to our construction, we write
\[ (4.6) \quad \sigma_{m-1}(\varepsilon) \leq \sigma. \]
And if
\[ (4.7) \quad s_{m-1}(\varepsilon) \leq \sigma + 2^{-i} \]
then, for \( m \leq j \leq k_h \), we get
\[ (4.8) \quad \sigma < s_j(\varepsilon) < \sigma + 2^{-i} + 2a_m. \]

Also, in the case
\[ (4.9) \quad s_{m-1}(\varepsilon) > \sigma + 2^{-i} \]
the relation \( (4.8) \) is still valid.
Now, if $s_{m-1}(\varepsilon) > \sigma + 2^{-i}$, then we are going to suppose that
\begin{equation}
(4.10) \quad s_{m-1}(\varepsilon) - (\sigma + 2^{-i}) < 2^{-i}.
\end{equation}
So, under this assumption, we can put the following
\begin{equation}
(4.11) \quad \sum_{j=m+1}^{k_h} a_j < 2^{-i} + a_m.
\end{equation}

**Proof.**

1°) Let $s_{m-1}(\varepsilon) \leq \sigma + 2^{-i}$. Since $\sigma < s_m(\varepsilon) \leq \sigma + 2^{-i} + a_m$, by (4.8), we can write
\[
\sigma < s_m(\varepsilon) - \sum_{j=m+1}^{k_h} a_j \leq \sigma + 2^{-i} + a_m.
\]
Therefore, we get
\[
\sigma + \sum_{j=m+1}^{k_h} a_j < s_m(\varepsilon) \leq \sigma + 2^{-i} + a_m
\]
and
\[
\sum_{j=m+1}^{k_h} a_j < 2^{-i} + a_m.
\]

2°) Let $s_{m-1}(\varepsilon) > \sigma + 2^{-i}$. Then $s_{m-1}(\varepsilon) = \sigma + 2^{-i} + \alpha$, where $0 < \alpha < 2^{-i}$. Therefore, we get
\[
s_{m-1}(\varepsilon) + a_m > \sigma + 2^{-i}
\]
\[
s_{m-1}(\varepsilon) + a_m - \sum_{j=m+1}^{k_h} a_j > \sigma + 2^{-i}
\]
\[
\sigma + 2^{-i} + \alpha + a_m - \sum_{j=m+1}^{k_h} a_j > \sigma + 2^{-i}
\]
or
\[
\sum_{j=m+1}^{k_h} a_j < \alpha + a_m.
\]
\[ \sum_{j=m+1}^{k_h} a_j < 2^{-1} + a_m. \]

This completes the proof of the Lemma.

Now, by the definition of \( \sigma_{k_h}(\varepsilon) \), we have

\[ \sigma_{k_h}(\varepsilon) = \frac{1}{k_h} \left[ (m-1) \sigma_{m-1} + \sum_{j=m}^{k_h} s_j(\varepsilon) \right] \]

and by (4.6) and (4.8)

\[ (4.12) \sigma_k(\varepsilon) < \sigma + \frac{1}{k_h} \left( 2^{-1} + 2a_m \right) \left( k_h - m + 1 \right). \]

If \( a_m \leq 2^{-1} \), then we easily get

\[ \sigma_{k_h}(\varepsilon) - \sigma < \eta/2. \]

If \( a_m > 2^{-1} \), then obviously \( m > n_i \). So, (4.1) and (4.11) give

\[ k_h - m < 2^{-1} \cdot \frac{m}{a_m} \]

and, we also have

\[ 1 < 2^{-1} \cdot \frac{m}{a_m}. \]

Therefore, from (4.12), we get

\[ \sigma_{k_h}(\varepsilon) - \sigma < \frac{1}{k_h} \left( 2^{-1} + 2a_m \right) \left( k_h - m + 1 \right) \]

\[ \sigma_{k_h}(\varepsilon) - \sigma < \frac{1}{k_h} \cdot 3a_m \cdot 2 \cdot 2^{-1} \cdot \frac{m}{a_m} \]

which implies, by (4.5), that

\[ \sigma_{k_h}(\varepsilon) - \sigma < \eta. \]

In a similar way, we can show that the difference between \( \sigma \) and minima \( \sigma_{j_h}(\varepsilon) \) tends to zero as \( h \rightarrow \infty \).
5. A Theorem For $\text{SC}(a_k) = \emptyset$.

In this chapter we are going to prove a theorem which gives a sufficient condition for $\text{SC}(a_k) = \emptyset$. This theorem will be based on Cauchy’s general convergence principle. It is known that, if the sequence

$$
\sigma_n(\varepsilon) = \frac{s_1(\varepsilon) + s_2(\varepsilon) + \ldots + s_n(\varepsilon)}{n}
$$

where $s_n(\varepsilon) = \sum_{v=1}^{n} \varepsilon_v a_v$, is divergent, then the series can not be $C_1$ — summable. So, what we need is having that, for at least one $k \geq 1$ and for each $n$

$$
| \sigma_{n+k}(\varepsilon) - \sigma_n(\varepsilon) | > \eta
$$

where $\eta > 0$.

Take $1 \leq k \leq n$, and write

$$
| \sigma_{n+k}(\varepsilon) - \sigma_n(\varepsilon) | =
$$

$$
= \frac{n[s_{n+1}(\varepsilon) + \ldots + s_{n+k}(\varepsilon)] - k[s_1(\varepsilon) + \ldots + s_n(\varepsilon)]}{n(n+k)}
$$

Using

$$
s_n(\varepsilon) = \sum_{v=1}^{n} \varepsilon_v a_v
$$

we get

$$
| \sigma_{n+k}(\varepsilon) - \sigma_n(\varepsilon) | =
$$

$$
= \left| \frac{k}{n(n+k)} \sum_{v=2}^{n+1} (v-1) \varepsilon_v a_v + \frac{1}{n+k} \sum_{v=1}^{k-1} (k-v)\varepsilon_{n+1+v} a_{n+1+v} \right|
$$

$$
= \left| \frac{k}{n(n+k)} \sum_{v=0}^{n+1} (n-v) \varepsilon_{n+1-v} a_{n+1-v} \right|
$$
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\[ + \frac{1}{n+k} \sum_{v=1}^{k-1} \varepsilon_{n+1+v} a_{n+1+v} \]

\[ = \frac{k}{n+k} \left[ \varepsilon_{n+1} a_{n+1} + \sum_{v=1}^{n-1} \varepsilon_{n+1-v} a_{n+1-v} \right] \]

\[ + \sum_{v=1}^{k-1} \varepsilon_{n+1+v} a_{n+1+v} \]

\[ = \frac{k}{n+k} \left[ \varepsilon_{n+1} a_{n+1} + \sum_{v=1}^{k-1} \varepsilon_{n+1-v} a_{n+1-v} \right] \]

\[ + \sum_{v=1}^{n-1} \varepsilon_{n+1-v} a_{n+1-v} \]

\[ + \sum_{v=1}^{k-1} \varepsilon_{n+1+v} a_{n+1+v} \]

\[ \geq \frac{k}{n+k} \left[ a_{n+1} - \sum_{v=1}^{k-1} \varepsilon_{n+1-v} a_{n+1-v} - \sum_{v=k}^{n-1} \varepsilon_{n+1-v} a_{n+1-v} \right] \]

\[ \geq \frac{k}{n+k} \left[ a_{n+1} - \sum_{|v|=1} a_{n+1+v} - \sum_{v=k}^{n-1} a_{n+1-v} \right] \]

\[ > \eta \]

So, we can express the following

**Theorem 5.1.** Let \( \Sigma a_k \) be a series of nonnegative terms. If there exists a number \( k \) \((1 \leq k \leq n)\) and an \( \eta > 0 \) such that for every \( n \) satisfying

\[ a_{n+1} - \sum_{|v|=1} a_{n+1+v} > (1 + \left( \frac{a}{k} \right)) \eta + \frac{n-1}{v=k} \]
REFERENCES


ÖZET

\[ \sum_{k=1}^{\infty} a_k \] reel ve negatif olmayan terimli bir nümerik sonsuz seri ve

\( (\epsilon) = \{\epsilon_k\}, \ (k = 1, 2, \ldots, \ \{\epsilon_k\} = \pm 1) \)

herhangi bir işaret dizisi olsun.

Verilen bir \((\epsilon)\) dizisi için \(\sum_{k=1}^{\infty} a_k\) serisinin \(n\)’inci kısmını topladım

\[ s_n(\epsilon) = \sum_{k=1}^{n} \epsilon_k a_k \]

ve \(n\)’inci kısmını \(C_1\) –topladım

\[ \sigma_n(\epsilon) = \frac{1}{n} \sum_{v=1}^{n} s_v(\epsilon) \]

ile belirttik ve \(\sigma_n(\epsilon)\)’un yakınsak olması halinde

\[ \sigma(\epsilon) = \lim_{n \to \infty} \sigma_n(\epsilon) \]

’a \(\sum a_k\) serisinin bir \(C_1\) –erişilir noktasi adını verdik. \(\sum a_k\) tüm \(C_1\) –erişilabilir noktalar cumlesini \(SC\) (\(a_k\)) ile gösterdik.

Bu araştırmamızda ise örnek olarak \(\sum a_k\) serisinin bütün \(C_1\) –erişilir noktaları cumlesi olan \(SC\) (\(a_k\)) cumlesini ele alıp \(SC\) (\(a_k\)) = \(R\) ve \(SC\) (\(a_k\)) = \(\emptyset\) olması hakkında bazı teoremler verdik, burada \(R\) reel sayılar cumlesini ve \(\emptyset\) ise boş cumleyi ifade etmektedir.
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