Characterizations of Spherical Curves in Euclidean n-Space

by

E. ÖZDAMAR – H. H. HACISALİHOĞLU

Faculté des Sciences de l'Université d'Ankara
Ankara, Turquie
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Characterizations of Spherical Curves in Euclidean n-Space

E. ÖZDAMAR  H. H. HACISALIHOĞLU

Department of Mathematics, Univ. of Ankara
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ABSTRACT:

We give the characterizations for the regular curves each of which lies on the \((n-1)\)
sphere \(S^{n-1}\) of \(n\) dimensional Euclidean Space \(E^n\). We express these characterizations
in the higher curvatures of the curves.

I. Basic Concepts.

**Theorem I.1:** If \(X\) is a parametrized curve in \(n\) dimensional
Euclidean space \(E^n\) then \(X\) can always be parametrized by an arc length parameter \([1]\).

Theorem I.1 says that, in general, we can have the arc-length
parametrized curve \(X (s)\) with arc-length parameter \(s\) as a para-
metrized curve in \(E^n\).

Let \(I\) be an open interval in the real line \(\mathbb{R}\). We shall inter-
pret this liberally to include not only the usual type of open interval
\(a < s < b\) (\(a, b\) real numbers), but also the types of \(a < s\) (a half
line to \(+ \infty\)), \(s < b\) (a half-line to \(- \infty\)), and also the whole real
line. Henceforth we denote an arc-length parametrized curve
of \(E^n\) by a map \(X: I \rightarrow E^n\) which is a \(C^\infty\) parametrization by arc-
length.

We assume that at each point \(X (s)\), of the curve \(X: I \rightarrow E^n\),
the derived vectors
\[
\{X', X'', \ldots, X^{(r)}\}
\]
are linearly independent, where
\[
X' = \frac{dX}{ds} (s), \quad X'' = \frac{d^2X}{ds^2} (s), \quad \ldots, \quad X^{(r)} = \frac{d^rX}{ds^r} (s).
\]
Therefore there exists an algorithm, called the Gramm-Schmidt process, for converting $X', X'', ..., X^{(r)}$ into an orthonormal basis
\[ \{V_1, V_2, \ldots, V_r\} \]
of the tangent space $T_{E^n}(X(s))$ of $E^n$ at the point $X(s) \in E^n$. This system is called the Frenet $r$-handed (or $r$-frame) of the curve $X$ at the point $X(s)$ [2].

If we denote the inner product (dot product) $E^n \times E^n \to |R$ over $E^n$ by $<, >$ we have
\[ <V_i, V_j> = \delta_{ij} \]
and then the derivatives of the frame vectors satisfy the following Frenet equations.

\[
\begin{align*}
V'_i & = -k_{i-1} V_{i-1} + k_i V_{i+1}, \quad 2 \leq i \leq r-1 \\
V'_1 & = k_1 V_2, \\
V'_r & = -k_{r-1} V_1,
\end{align*}
\]

where $k_i, 1 \leq i \leq r-1$, is the curvature, with order $i$, of the curve $X$ at its point $X(s)$ [2]. These formulae (I.1) are called the Frenet Formulae which give us the derived vectors $V'_i, 1 \leq i \leq r$. Then we mention the following theorem which is important in Chapter III.

**Theorem I.2**: Let $X: I \to E^n$ be a regular curve. At the point $X(s)$ of it if Frenet $n$-frame is
\[ \{V_1, V_2, \ldots, V_n\} \]
then we have
\[ X^{(p)} = \sum_{j=1}^{p} a_j V_j, \quad 1 \leq p \leq n, \]
where $a_j \in |R$.

**Proof**: We use the induction process:

(i) Since the curve is given by its arc-length parameter $s$, $X' = V_1$. Therefore if $p = 1$ the theorem is trivial.

(ii) Let us suppose that the theorem is proved for the cases
$1 \leq r < n$. Then we prove that the theorem is also valid for the case $p = r+1$.

Since we can write

$$X^{(r)} = \sum_{j=1}^{r} a_j V_j$$

differentiating this, with respect to $s$, we obtain

$$X^{(r+1)} = \sum_{j=1}^{r} a_j' V_j + \sum_{j=1}^{r} a_j V_j'$$

Using Equations (1.1) this gives us

$$X^{(r+1)} = \sum_{j=1}^{r} \left[a_j' V_j + a_j (-k_{j-1} V_{j-1} + k_j V_{j+1})\right]$$

where if we write that

$$b_1 = a_1' - k_1 a_2,$$
$$b_j = a_j' + a_{j-1} k_{j-1} - a_{j+1} k_j \quad 2 \leq j \leq r-1,$$
$$b_r = a_r' + a_{r-1} k_{r-1},$$
$$b_{r+1} = a_r k_r,$$

then we obtain

$$X^{(r+1)} = \sum_{j=1}^{r+1} b_j V_j$$

which completes the theorem.

II. Osculating $p$-Spheres $S^p$ and The Curvature Lines.

Definition II.1: The $p$-sphere $S^p$ in $E^n$ which passes through $X(s)$ and is in contact with the curve having $p+2$ points in common at the point of contact $X(s)$ is called the osculating $p$-sphere to the curve at $X(s)$.

At any given point, the curve has exactly $p$-th order contact with its osculating $p$-sphere for $p = r$. 
We suppose that at the point $X(s)$ of the curve $k_{n-1} \neq 0$ and then we will educate the osculating sphere $S^{n-2}$. In order to do this we will need the following theorem.

**Theorem II.1**: Let $k_{n-1} \neq 0$ at every point $X(s)$ of a curve $X: I \rightarrow E^n$. Then at the point $X(s)$ of the curve the center of $(n-2)$-osculating sphere $S^{n-2}$ is

$$a = X - \sum_{i=1}^{n-2} m_i V_i + \lambda V_n, \quad a \in E^n,$$

where $\lambda \in \mathbb{R}$ and $m_1 = 0, m_2 = -1/k_1$ and

$$m_i = \{ m'_{i-1} + m_{i-2} k_{i-2} \} \frac{1}{k_{i-1}}, \quad 2 < i < n.$$

**Proof**: Suppose that at the point $X(s)$ there is at least one $(n-2)$-osculating sphere with the center $a \in E^n$ and radius $r \in \mathbb{R}$. In this case let define the function $f: I \rightarrow \mathbb{R}$ as

$$f(s) = \langle X(s) - a, X(s) - a \rangle - r^2.$$  

Since $X(s) \in S^{n-2}$, we have

$$f(s) = 0.$$

On the other hand, if $X(s)$ is a second order contact point of the curve and the osculating sphere $S^{n-2}$ for the case $\forall s_j \rightarrow s$ then we have

$$f(s_1) = 0$$

$$f(s_2) = 0,$$

where $s_1, s_2 \in I$. Applying the mean value theorem to these equations we obtain $f(s) = 0$ and $f'(s) = 0$. Similarly, if $X(s)$ is a $n$-th order point of the curve and the osculating sphere $S^{n-2}$ for the case $\forall s_j \rightarrow s$

we have

$$f(s_j) = 0, \quad 1 \leq j \leq n; \quad s_j, s \in I$$

$$f(s) = 0,$$

and from the mean value theorem

$$f(s) = 0, \quad f'(s) = 0, ..., \quad f^{(n-1)}(s) = 0.$$
Since $k_{n-1} \neq 0$ we can imagine that Frenet $n$-frame is exist at the point $X(s)$ of the curve. Hence, \{$V_1, V_2, \ldots, V_n$\} is a basis of tangent space $T_{F^n}(X(s))$ and $(X(s)-a) \in T_{F^n}(X(s))$ can be expressed, in a unique way, as

$$X(s)-a = \sum_{i=1}^{n} m_i V_i, \\forall m_i \in \mathbb{R}.$$  

Replacing (II.1) in (II.2) we obtain

$$f'(s) = 2 \langle X'(s), X(s)-a \rangle = 0.$$  

On the other hand $s$ is arc-length parameter and so $V_1 = X'$. Hence

$$\langle V_1, X(s)-a \rangle = 0.$$  

and then $m_1 = 0$ so

$$X(s)-a = \sum_{i=2}^{n} m_i V_i.$$  

In Equation (II.1) since $f''(s) = 0$ and using the Frenet formulae we obtain that

$$\frac{1}{2} f''(s) = \frac{d}{ds} (\langle V_1, X(s)-a \rangle) = 0$$

and then

$$\langle k_1 V_2, X(s)-a \rangle + \langle V_1, V_1 \rangle = 0$$

$$k_1 \langle V_2, X(s)-a \rangle + 1 = 0$$

$$m_2 = -1/k_1.$$  

Hence the theorem is proved for the coefficients $m_i$.

From the Equations (II.2) one can write

$$f'''(s) = 2 \langle X'''(s), X(s)-a \rangle = 0.$$  

On the other hand differentiating the equation

$$X' = V_1$$

according to $s$ and using the Frenet Formulae we obtain that

$$X'' = k_1 V_2$$

$$X''' = -k_1^2 V_1 + k_1' V_2 + k_1 k_2 V_3.$$
Replacing the last equation in \( f'''(s) = 0 \) and using (II.3) we have

\[
- k_1^2 V_1 + k'_1 V_2 + k_1 k_2 V_3, \quad \sum_{i=2}^{n} m_i V_i > = 0
\]

or

\[
k'_1 m_2 + k_1 k_2 m_3 = 0.
\]

Since \( m_2 = -1/k_1 \) for \( m_3 \) we can have

\[
m_3 = m'_2 / k_2
\]

and the theorem is also proved for the case \( i = 3 \).

Suppose that the theorem is valid for the cases for \( p \) such that \( 2 < p < n-1 \) and then we will prove it for the case \( p+1 \).

Let define a function \( \Psi_p(s) \) by the equation

\[
(II.3) \quad f^{(p)}(s) = < X^{(p)}(s), X(s) - a > + \Psi_p(s).
\]

From the derivative of

\[
f'(s) = 2 < X'(s), X(s) - a >
\]

we can see that in the expression of \( f^{(p)}(s) \) the derivatives higher than \( X^{(p)}(s) \) do not exist. From the Theorem (I.2) we can write

\[
(II.4) \quad X^{(p)}(s) = \sum_{j=1}^{p} a_j V_j, \quad \forall a_j \in \mathbb{R}.
\]

Replacing (II.3) and (II.4) in the Equation \( f^{(p)}(s) = 0 \) we have

\[
< \sum_{j=1}^{p} a_j V_j, \sum_{j=2}^{n} m_j V_j > + \Psi_p(s) = 0
\]

or

\[
\sum_{j=2}^{p} m_j a_j + \Psi_p(s) = 0.
\]

Therefore we have

\[
(II.5) \quad m_p = - \frac{1}{a_p} \left[ \sum_{j=2}^{p-1} m_j a_j + \Psi_p(s) \right].
\]

Differentiating the Equations (II.3) and (II.4) we have, respectively,
$f^{(p+1)} (s) = < X^{(p+1)} (s), X (s) - a >$

$+ < X^{(p)} (s), V_1 > + \Psi'_p (s) = 0$

and

$X^{(p+1)} (s) = (a'_1 - k_1 a_2) V_1 + \sum_{j=2}^{p-1} (a'_j + a_{j-1} k_{j-1} - a_{j+1} k_j) V_j$

$+ (a'_p + a_{p-1} k_{p-1}) V_p + a_p k_p V_{p+1}$.

Hence replacing the values of $X^{(p+1)} (s)$ and $X(s) - a$ in $f^{(p+1)} (s) = 0$ we have

$\sum_{j=2}^{p-1} (a'_j + a_{j-1} k_{j-1} - a_{j+1} k_j) m_j + (a'_p + a_{p-1} k_{p-1}) m_p$

$+ a_p k_p m_{p+1} + a_1 + \Psi'_p (s) = 0$

From the last equation, calculation gives us that the value of $m_{p+1}$ is

$m_{p+1} = - \frac{1}{a_p k_p} \{ \sum_{j=2}^{p-1} (a'_j + a_{j-1} k_{j-1} - a_{j+1} k_j) m_j$

$+ (a'_p + a_{p-1} k_{p-1}) m_p + a_1 + \Psi'_p (s) \}$.

And from the Equation (II.5) differentiation gives us that

$m'_p = - \frac{1}{a_p} \{ \sum_{j=2}^{p-1} (a'_j + a_{j-1} k_{j-1} + a_{j+1} k_j) m_j$

$+(a'_p + a_{p-2} k_{p-2}) m_{p-1} + (a'_p + a_{p-1} k_{p-1}) m_p + a_1 + \Psi'_p (s) \}$.

Hence we can write

$(II.6) \quad m_{p+1} = \{ m'_p + m_{p-1} k_{p-1} \} \frac{1}{k_p}$

and since we obtain $m_{p+1}$ from the equation $f^{(p+1)}(s) = 0$ the coefficients $m_2, m_3, \ldots, m_{n-1}$ are determinant in an equation like (II.6). $m_n = \lambda \in \mathbb{R}$ is a parameter. ■

**Corollary I:** At any point $X(s)$ of a regular curve $X:I \to E^n$ if $k_{n-1} \neq 0$ then all the centers of $(n-2)$ osculating spheres $S^{n-2}$ are collinear.
Proof: From the Theorem (II.1) the center of $S^{n-2}$ at $X(s)$ is

$$a = X(s) - \sum_{i=2}^{n-1} m_i V_i - \lambda V_n, \quad \lambda \in \mathbb{R},$$

where,

$$m_2 = -1/k_1, \quad m_i = \left\{m'_{i-1} + m_{i-2}k_{i-2}\right\} \frac{1}{k_{i-1}}.$$

For $\forall s \in I$, $m_i, V_i, X(s)$ are constant. Hence $a$, the center of $(n-2)$-osculating sphere $S^{n-2}$, lies on the straight line which passes through the point $X(s) - \sum_{i=2}^{n-1} m_i V_i$ and parallel to the vector $V_n$. ■

Definition (II.2): Let $X: I \rightarrow \mathbb{E}^n$ be a given regular curve. The locus of the centers of $(n-2)$-osculating skteres $S^{n-2}$ is called the curvature line of the curve, at the point $X(s)$.

Corollary II. At any point $X(s)$ of a regular curve $X: I \rightarrow \mathbb{E}^n$ if $k_{n-1} \neq 0$ then the center of $(n-1)$-osculating sphere $S^{n-1}$ is

$$a = X(s) - \sum_{i=2}^{n} m_i V_i$$

where, $m_1 = 0$, $m_2 = -1/k_1$ and

$$m_i = \left\{m'_{i-1} + m_{i-2}k_{i-2}\right\} \frac{1}{k_{i-1}}, \quad 2 < i \leq n.$$

Proof: According to Definition (II.1) the point $X(s)$ is a $(n+1)$-th order contact point of the curve $X$ and its $(n-1)$-osculating sphere. Therefore we have the expressions (II.2) and also $f^{(n)}(s) = 0$. Hence we can repeat here the same proof of Theorem (II.1). ■

Corollary III. If $\forall k_{n-1} \neq 0, s \in I$, for the curve $X: I \rightarrow \mathbb{E}^n$ then at the point $X(s)$ the osculating sphere $S^{n-1}$ is unique and its radius is $r = (\sum_{i=2}^{n} m_i^2)^{1/2}$.

Proof: According to Corollary II the center of $(n-1)$-osculating sphere $S^{n-1}$ is unique so $S^{n-1}$ is unique.
At the point \( X (s) \) the radius \( r \) of osculating sphere \( S^{n-1} \) is
\[
r = \| a - X (s) \|.
\]
From the corollary II we have
\[
r = \| a - X (s) \| = \| \sum_{i=1}^{n} m_i V_i \|
\]
\[
= \left( \sum_{i=1}^{n} m_i^2 \right)^{1/2}.
\]

**III. Spherical Curve of \( E^n \) and Its Characterization.**

In this paragraph we will give a necessary and sufficient condition for a curve of \( E^n \) to be a spherical curve.

**Definition III. 1:** Let \( X: I \to E^n \) be a curve and \( S^p \subset E^n \) be a \( p \)-sphere. If \( X \subset S^p \) then \( X \) is called a spherical curve in \( E^n \).

The case \( p=n-1 \) is supposed in this paragraph. Because of \( S^p = S^{n-1} \cap H_{n(p+1)} \) every curve \( X \) of \( S^p \) in \( E^n \) lies in a \((p+1)\)-hyperplane \( H_{n(p+1)} \) [3] so this case is not a special case. Since a \((p+1)\)-hyperplane is isomorphic to Euclidean \((p+1)\)-space \( E^{p+1} \), a curve of \( S^p \) can be taken as another curve of another sphere \( S^p_0 \subset E^{p+1} \). Hence \( X \subset S^p \subset E^n \) so we can see that \( k_{p+1} \neq 0 \). Therefore we can only suppose the curves of \( S^{n-1} \) whose curvature \( k_{n-1} \neq 0 \).

**Theorem III. 1:** Let \( X: I \to E^n \) be a regular curve such that
\[
k_{n-1} \neq 0, \forall s \in I, m_1 = 0, m_2 = -\frac{1}{k_1}
\]
\[
m_i = \{ m'_{i-1} + m_{i-3} k_{i-2} \} \cdot \frac{1}{k_{i-1}}, 2 < i \leq n,
\]
and \( X \subset S_0^{n-1} \); where \( S_0^{n-1} \) is an \((n-1)\)-sphere with the center \( O \). Then
\[
< X (s), V_i > = m_i
\]
where \( \{ V_1, V_2, \ldots, V_n \} \) is the Frenet \( n \)-frame at the point \( X (s) \) of the curve.
Proof: We apply the induction process:

If \( i = 2 \) and the radius of \( S^n_0 \) is \( r \) we can write

\[
< X(s), X(s) > = r^2
\]

and then from this by differentiation, with respect to \( s \),

\[
< X(s), X'(s) > = 0
\]

or

\[
< X''(s), X(s) > + < X'(s), X'(s) > = 0
\]

or

\[
< X''(s), X(s) > + 1 = 0.
\]

On the other hand since we know that \( V_2 = X''(s) / \| X''(s) \| = k_1 \) [2] we can have

\[
k_1 < V_2, X(s) > = -1
\]

or

\[
< V_2, X(s) > = -1 / k_1 = m_2
\]

which proves the theorem in the case \( i = 2 \).

Suppose that the theorem is proved in the cases \( p < n \). Then we can write

\[
< X(s), V_p > = m_p
\]

which gives us, by differentiation, with respect to \( s \),

\[
< V_1, V_p > + < X(s), V'_p > = m'_p
\]

in this last equation, replacing the Frenet Formulae (I,1) we have

\[
< X(s), V_{p+1} > = \{ m'_p + m_{p-1}k_{p-1} \} \frac{1}{k_p}
\]

\[
< X(s), V_{p+1} > = m_{p+1}
\]

which completes the theorem. ■

Theorem III. 2: Let \( X: I \to \mathbb{E}^n \) be a given regular curve such that \( k_{n-1} \neq 0 \), \( \forall s \in I \). If \( X \subset S^{n-1}_0 \) then all the \((n-1)\)-osculating spheres of the curve \( X \) coincide with \( S^{n-1}_0 \).

Proof: Suppose that the center of \((n-1)\)-osculating sphere at the point \( X(s) \) of \( X \) is \( a \). From the Corollary II of Theorem II.1 we have
\[ a = X(s) - \sum_{j=2}^{n} m_j V_j \]

where \( m_1 = 0, m_2 = -1/k_1, m_i = \left\{ m'_{i-1} + m_{i-2} k_{i-2} \right\} \frac{1}{k_i} \),

\[ 2 < i \leq n \] and \( \{V_1, V_2, ..., V_n\} \) is the Frenet \( n \)-frame at \( X(s) \) of \( X \). According to Theorem III.1 the expression of \( a \) can be write as

\[ a = X(s) - \sum_{j=1}^{n} < X(s), V_j > V_j. \]

Since \( \{V_1, V_2, ..., V_n\} \) is a basis of the tangent space \( T_{E^n}(O) \) we can have

\[ X(s) = \sum_{j=1}^{n} < X(s), V_j > V_j \]

and then

\[ a = X(s) - X(s) \]

or

\[ a = 0 \]

which shows that the centers of \( S_0^{n-1} \) and \( (n-1) \)-osculating sphere at \( X(s) \) of \( X \) coincide. On the other hand since \( d(X(s), O) = r \) we see that the theorem is completed. \( \blacksquare \)

**Corollary I.** If \( S_b^{n-1} \subset E^n \) is an \( (n-1) \)-sphere and the curve \( X \) is \( X \subset S_b^{n-1} \) then \( (n-1) \)-osculating sphere at the point \( X(s) \) of \( X \) is \( S_b^{n-1} \).

The proof of this corollary can be given in the light of the fact that “The spheres with the same radius are isometric”.

The radius of an \( (n-1) \)-osculating sphere of a curve \( X \) depends on the center \( X(s) \) of the sphere. The following theorem makes clear this dependence.

**Theorem III.3:** Let \( X: I \rightarrow E^n \) be a given regular curve whose \( k_{n-1} \neq 0 \) for \( \forall s \in I \) and let \( m_n \neq 0 \) (see the Corollary II of Theorem II.1). The radii of \( (n-1) \)-osculating spheres at \( X(s) \) of \( X \) are constant for \( \forall s \in I \) \( \Leftrightarrow \) The centers of the \( (n-1) \)-osculating spheres are the same point.

**Proof:** We need the following calculation:
As we know, let a (s) and r (s) be, respectively, the center and the radius of (n-1)-osculating sphere at X (s) of the curve X. Since X (s) is a point of the (n-1)-osculating sphere we have
\[ < X (s) - a (s), X (s) - a (s) > = (r (s))^2 \]
which gives us, by differentiation with respect to s,

\[ (III.1) \quad < V_1, X (s) - a (s) > = - \frac{da}{ds} (s), X (s) - a (s) > = r (s). \frac{dr}{ds} (s). \]

According to Corollary II of Theorem II.1 we have
\[ < V_1, X (s) - a (s) > = 0 \]
and replacing this in (III.1) we obtain

\[ (III.2) \quad \frac{da}{ds} (s), X (s) - a (s) > = - r (s). \frac{dr}{ds} (s). \]

Now we can give the proof:

(Necessity): Suppose that at every point X (s) of the curve X the radii r (s) are constant. According to Corollary II of Theorem II.1 the radius of (n-1)-osculating sphere at X (s) is

\[ r (s) = (\sum_{i=2}^{n} m_i^2)^{1/2} \]

\[ r (s) = \text{constant} \Rightarrow \frac{dr}{ds} = 0 \]

and

\[ (III.3) \quad \sum_{i=2}^{n} m_i m_i^\prime = 0. \]

From the corollary II of Theorem II.1 since we have
\[ m_i = - k_{i-1} m_{i-1} + k_i m_{i+1}, \quad 2 < i \leq n-1. \]

Equation (III.3) reduces to

\[ (III.4) \quad m_2 m_2^\prime + \sum_{i=3}^{n-1} m_i [ - k_{i-1} m_{i-1} + k_i m_{i+1} ] + m_n m_n^\prime = 0 \]

or replacing m_2 = m_3 k_2 in (III.4)
or after some cancellations

\[(\text{III.5}) \quad m_n (m'_n + k_{n-1}m_{n-1}) = 0.\]

Then according to Theorem II.1 we have

\[\frac{da}{ds} (s) = V_1 - \sum_{i=2}^{n} m'_i V_i - \sum_{i=2}^{n} m_i V'_i\]

where replacing (I.1) and (III.4) we obtain

\[(\text{III.6}) \quad \frac{da}{ds} = (m'_n + k_{n-1}m_{n-1}) V_n.\]

From the Equations (III.5) and (III.6)

\[\frac{da}{ds} = 0, \quad \forall s \in I\]

and so

\[a (s) = \text{constant}.\]

(Sufficiency): Suppose that \(a (s) = \text{constant}, \forall s \in I.\) Then \(\frac{da}{ds} = 0\)

and according to (III.2)

\[< \frac{da}{ds}, X (s) - a (s) > = - r (s) \frac{dr}{ds} = 0\]

or

\[r (s) \frac{dr}{ds} (s) = 0.\]

In the last equation if \(r (s) = 0\) then from Corollary III of Theorem II.1 we have

\[\sum_{i=2}^{n} m_i = 0\]

which gives us \(m_i = 0, \quad 2 \leq i \leq n.\)

On the other hand

\[m_2 = - 1/k_1.\]

Since \(k_i = \| X'' (s) \| / \| X' (s) \| [2]\) the case \(m_2 = 0\) implies that \(\| X' (s) \| = 0\) or \(\| X'' (s) \| \to \infty.\) In the case \(\| X' (s) \| = 0\)
the curve is not regular. Then we must have $\|X'(s)\| \neq 0$ and in $E^n$ $\|X''(s)\|$ can not be infinitive. Therefore we must have

$$\frac{dr}{ds}(s) = 0$$

and so $r(s) = \text{constant}$. 

A characterization of a curve in $E^n$ to be an $(n-1)$-sphere can be given by the following theorem.

**Theorem III.4:** Let $X: I \rightarrow E^n$ be a regular curve such that $k_{n-1} \neq 0$, $\forall s \in I$ and $m_n(s) \neq 0$. The curve $X$ lies on a $(n-1)$-sphere $\Leftrightarrow$ The centers of $(n-1)$-osculating spheres of the curve $X$ are all the same point.

**Proof:** (Necessity): Suppose that $X$ lies on $S_b^{n-1}$. Then according to Corollary I of Theorem III.2, for $\forall s \in I$ at the points $X(s)$ of $X$, $(n-1)$-osculating sphere is $S_b^{n-1}$ whose center is a fixed point.

(Sufficiency): Suppose that, at the point $X(s)$, the center of $(n-1)$-osculating sphere of $X$ is a fixed point $b$. Then Theorem III.3 says that, at every point $X(s)$ of $X$, the radii of $(n-1)$-osculating spheres are also equal. Hence for $\forall s \in I$ at every point $X(s)$,

$$d(X(s), b) = r = \text{constant}$$

which means that $X$ is a spherical curve. For $\forall s \in I$ the curvature $k_{n-1} \neq 0$ implies that this sphere is $S_b^{n-1}$. 

Another characterization of a curve in $E^n$ to be on a $(n-1)$-sphere can be given in terms of its curvatures by the following theorem.

**Theorem III.5:** Let $X: I \rightarrow E^n$ be a regular curve such that for $\forall s \in I$, $k_{n-1} \neq 0$, $m_n \neq 0$, $m_1 = 0$, $m_2 = -1/k_1$ and for $2 < i \leq n$,

$$m_i = \left( m'_{i-1} + m_{i-1}k_{i-2} \right) \frac{1}{k_{i-1}}.$$  

The curve $X$ lies on a sphere $S^{n-1} \Leftrightarrow m'_n + m_{n-1}k_{n-1} = 0$. 

Proof: (Necessity): According to Corollary II of Theorem II.1 the center of \((n-1)\)-osculating sphere at \(X(s)\) is

\[
(III.7) \quad a(s) = X(s) - \sum_{j=2}^{n} m_j V_j.
\]

On the other hand according to Theorem III.4 it is necessary that \(a(s)\) is a fixed point for the curve \(X\) to lie on a \((n-1)\)-sphere. This implies that

\[\frac{da}{ds} = 0.\]

Hence from Equation (III.7), by differentiation with respect to \(s\), we have

\[
(III.8) \quad \frac{da}{ds} = (m'_n + m_{n-1} k_{n-1}) V_n = 0
\]

which completes the necessity of the theorem.

(Sufficiency): Suppose that for a curve \(X\) we have

\[m'_n + m_{n-1} k_{n-1} = 0.\]

Replacing this in (III.8) we obtain

\[\frac{da}{ds} = 0\]

which implies that

\[a(s) = \text{constant}.\]

Thus jointing this result with Theorem III.4 we see that \(X\) is a spherical curve and for \(\forall s \in \mathbb{I}\), since \(k_{n-1} \neq 0\) this lies on a \((n-1)\)-sphere of \(E^n\). \(\blacksquare\)

IV. Special Cases.

1. The Case \(n=3\).

In the case that \(n=3\) the formulae in Theorem III.5 reduces to

\[
(IV.1) \quad m'_3 + k_2 m_2 = 0,
\]

where replacing \(m_3 = m'_2 / k_2\) we have

\[\frac{(m'_2 / k_2)'}{k_2} + m_2 k_2 = 0.\]

Since \(m_2 = -1 / k_1\) the last equation gives us
(IV.2) \[ \frac{1}{k_1} k_2 + \left[ \left( \frac{1}{k_1} \right)' \frac{1}{k_2} \right]' = 0, \]

where replacing
\[ \frac{1}{k_1} = \theta, \quad \frac{1}{k_2} = \sigma \text{ and } k_2 = \tau \]
we obtain
(IV.3) \[ \theta \tau' + (\theta' \sigma)' = 0 \]

which is well-known, in the books on elementary differential geometry, characterization for spherical curves.

On the other hand in the case \( n = 3 \) the function \( f \) in [4] can be taken as \( f = -m_3 \). Similarly, taking \( n = 3 \) in Corollary III of Theorem III.1, the radius of the sphere can be obtained as

\[ r = (m_2^2 + m_3^2)^{1/2} \]
or
\[ r = \left[ \left( \frac{1}{k_1} \right)^2 + f_2 \right]^{1/2} \]

which is the same value in [4]. Hence, for \( n = 3 \), Theorem I.1 in [4], can be obtained as another special case from Theorem III.5. Since \( f = -m_3 \) are, respectively,

\[ m_3 k_2 = m_2', \quad m_3' + k_2 m_2 = 0 \]

which can be obtained from Theorem III.5, for \( n = 3 \).

On the other hand since Theorem 1.2 in [4] is deduced from (IV.2) we can say that it is also another special case of the Theorem III.5.
REFERENCES


Özet:

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