A CHARACTERIZATION OF CYLINDRICAL HELIX STRIP

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Abstract. In this paper, we investigate cylindrical helix strips. We give a new definition and a characterization of cylindrical helix strip. We use some characterizations of general helix and the Terquem theorem (one of the Joachimsthal Theorems for constant distances between two surfaces).

1. Introduction

In 3-dimensional Euclidean Space, a regular curve is described by its curvatures $k_1$ and $k_2$ and also a strip is described by its curvatures $k_n$, $k_g$ and $t_r$. The relations between the curvatures of a strip and the curvatures of the curve can be seen in many differential books and papers. We know that a regular curve is called a general helix if its first and second curvatures $k_1$ and $k_2$ are not constant, but $\frac{k_1}{k_2}$ is constant ([2], [7]). Also if a helix lie on a cylinder, it is called a cylindrical helix and a cylindrical helix has the strip at $\alpha(s)$. The cylindrical helix strips provide being a helix condition and cylindrical helix condition at the point $\alpha(s)$ of the strip by using the curvatures of helix $k_1$ and $k_2$.

2. Preliminaries

2.1. The Theory of the Curves.

Definition 2.1. If $\alpha : I \subset \mathbb{R} \rightarrow E^n$ is a smooth transformation, then $\alpha$ is called a curve (from the class of $C^\infty$). Here $I$ is an open interval of $\mathbb{R}$ ([11]).
Definition 2.2. Let the curve $\alpha \subset E^n$ be a regular curve coordinate neigbourhood and $\{V_1(s), V_2(s), ..., V_r(s)\}$ be the Frenet frame at the point $\alpha(s)$ that correspond for every $s \in I$. Accordingly,

$$k_i : \ I \rightarrow R \quad s \rightarrow k_i(s) = \langle V_i(s), V_{i+1}(s) \rangle.$$  

We know that the function $k_i$ is called $i$-th curvature function of the curve and the real number $k_i(s)$ is called $i$-th curvature of the curve for each $s \in I$ ([2]). The relation between the derivatives of the Frenet vectors among $\alpha$ and the curvatures are given with a theorem as follows:

Definition 2.3. Let $M \subset E^n$ be the curve with neigbouring $(I, \alpha)$. Let $s \in I$ be arc parameter. If $k_i(s)$ and $\{V_1(s), V_2(s), ..., V_r(s)\}$ be the $i$-th curvature and the Frenet r-frame at the point $\alpha(s)$, then

$$
\begin{align*}
\text{i. } V_1'(s) &= k_1(s)V_2(s) \\
\text{ii. } V_i'(s) &= -k_{i-1}(s)V_{i-1}(s) + k_i(s)V_{i+1}(s), \quad \ldots \quad 1 \leq i \leq r, \\
\text{iii. } V_r'(s) &= -k_{r-1}(s)V_{r-1}(s)
\end{align*}
$$

([2]).

The equations that about the covariant derivatives of the Frenet r-frame $\{V_1(s), V_2(s), ..., V_r(s)\}$ the Frenet vectors $V_i(s)$ along the curve can be written as

$$
\begin{bmatrix}
V_1(s) \\
V_2(s) \\
V_3(s) \\
\vdots \\
V_{r-2}(s) \\
V_{r-1}(s) \\
V_r(s)
\end{bmatrix}
= 
\begin{bmatrix}
0 & k_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-k_1 & 0 & k_2 & 0 & \cdots & 0 & 0 & 0 \\
0 & -k_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & k_{r-2} & 0 \\
0 & 0 & 0 & 0 & \cdots & -k_{r-2} & 0 & k_{r-1} \\
0 & 0 & 0 & 0 & \cdots & 0 & -k_{r-1} & 0
\end{bmatrix}
\begin{bmatrix}
V_1(s) \\
V_2(s) \\
V_3(s) \\
\vdots \\
V_{r-2}(s) \\
V_{r-1}(s) \\
V_r(s)
\end{bmatrix}
$$
These formulas are called Frenet Formulas ([2]).

In special case if we take \( n = 3 \) above the last matrix equations, we obtain following matrix the equation

\[
\begin{bmatrix}
V_1' \\
V_2' \\
V_3'
\end{bmatrix} = \begin{bmatrix}
0 & k_1 & 0 \\
-k_1 & 0 & k_2 \\
0 & -k_2 & 0
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}
or
\begin{bmatrix}
n' \\
b'
\end{bmatrix} = \begin{bmatrix}
-k_1 & 0 & k_2 \\
0 & -k_2 & 0
\end{bmatrix}
\begin{bmatrix}
t \\
n
\end{bmatrix}
\]

The first curvature of the curve \( k_1(s) \) is called only curvature and the second curvature of the curve \( k_2(s) \) is known as torsion ([2]).

If the Frenet vectors are shown as \( V_1 = t, V_2 = n, V_3 = b \) in \( E^3 \), and the curvatures of the curve are shown as \( k_1 = \kappa \) and \( k_2 = \tau \),

\[
\begin{bmatrix}
t' \\
n' \\
b'
\end{bmatrix} = \begin{bmatrix}
0 & \kappa & 0 \\
-\kappa & 0 & \tau \\
0 & \tau & 0
\end{bmatrix}
\begin{bmatrix}
t \\
n \\
b
\end{bmatrix}
\]

or the equations are as follows,

\[
\begin{align*}
t' &= \kappa n \\
n' &= -\kappa t + \tau b \\
b' &= -\tau n.
\end{align*}
\]

2.2. The Strip Theory.

**Definition 2.4.** Let \( M \) and \( \alpha \) be a surface in \( E^3 \) and a curve in \( M \subset E^3 \). We define a surface element of \( M \) is the part of a tangent plane at the neighbour of the point. The locus of these surface element along the curve \( \alpha \) is called a strip or
curve-surface pair and is shown as $(\alpha, M)$.

\[ \frac{\vec{t}}{t}, \frac{\vec{n}}{n}, \frac{\vec{b}}{b} \] is called Frenet Frame or Frenet Trehold. Also Frenet vectors of the curve is shown as $\{V_1, V_2, V_3\}$. In here $V_1 = \vec{t}$, $V_2 = \vec{n}$, $V_3 = \vec{b}$.

Let $\vec{t}$ be the tangent vector field of the curve $\alpha$, $\vec{n}$ be the normal vector field of the curve $\alpha$ and $\vec{b}$ be the binormal vector field of the curve $\alpha$.

\[ \alpha : I \subset M \rightarrow E^3 \quad s \rightarrow \alpha(s). \]

If $\alpha : I \rightarrow E^3$ is a curve in $E^3$ with $\|\alpha(s)\| = 1$, then $\alpha$ is called unit velocity. Let $s \in I$ be the arc length parameter of $\alpha$. In $E^3$ for a curve $\alpha$ with unit velocity, $\{\vec{t}, \vec{n}, \vec{b}\}$ Frenet vector fields are calculated as follows ([2])

\[ \begin{align*}
\vec{t} &= \alpha'(s), \\
\vec{n} &= \frac{\alpha''(s)}{\|\alpha''(s)\|}, \\
\vec{b} &= \vec{t} \times \vec{n}.
\end{align*} \]
Strip vector fields of a strip which belong to the curve \( \alpha \) are \( \{ \xi, \eta, \zeta \} \). These vector fields are:

- Strip tangent vector field is \( \overrightarrow{t} = \xi \)
- Strip normal vector field is \( \overrightarrow{n} = \zeta \)
- Strip binormal vector field is \( \overrightarrow{n} = \xi \Lambda \xi \) \( \text{([6])} \).

\[ \xi = \frac{c \eta - b \zeta}{a}, \quad \eta = -c \xi + a \zeta, \quad \zeta = b \xi - a \eta \] \( \text{([1])} \)
2.5. Some Relations between Frenet Vector Fields of a Curve and Strip Vector Fields of a Strip. Let \( \{ \xi, \eta, \zeta \} \), \( \{ \overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b} \} \) and \( \phi \) be the unit strip vector fields, the unit Frenet vector fields and the angle between \( \overrightarrow{\eta} \) and \( \overrightarrow{n} \) on \( \alpha \).

\begin{align*}
\langle \overrightarrow{t}, \overrightarrow{\xi} \rangle &= 0 \\
\langle \overrightarrow{t}, \overrightarrow{\eta} \rangle &= 0 \\
\langle \overrightarrow{t}, \overrightarrow{\zeta} \rangle &= 0 \\
\langle \overrightarrow{t}, \overrightarrow{\eta} \rangle &= 0.
\end{align*}

2.5.1. The Equations of the Strip Vector Fields in type of the Frenet vector Fields. Let \( \{ \overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b} \} \), \( \{ \xi, \eta, \zeta \} \) and \( \phi \) be the Frenet Vector fields, strip vector fields and the angle between \( \overrightarrow{\eta} \) and \( \overrightarrow{n} \). We can write the following equations by the Figure 4.

\begin{align*}
\overrightarrow{\xi} &= \overrightarrow{t} \\
\overrightarrow{\eta} &= \cos \phi \overrightarrow{n} - \sin \phi \overrightarrow{b} \\
\overrightarrow{\zeta} &= \sin \phi \overrightarrow{n} + \cos \phi \overrightarrow{b}.
\end{align*}
or in matrix form
\[
\begin{bmatrix}
\xi \\
\eta \\
\zeta
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \varphi & -\sin \varphi \\
0 & \sin \varphi & \cos \varphi
\end{bmatrix}
\begin{bmatrix}
\vec{t} \\
\vec{n} \\
\vec{b}
\end{bmatrix}.
\]

2.5.2. The Equations of the Frenet Vector Fields in type of the Strip Vector Fields.
By the help of the Figure 4 we can write
\[
\begin{align*}
\vec{t} &= \xi \\
\vec{n} &= \cos \varphi \eta + \sin \varphi \zeta \\
\vec{b} &= -\sin \varphi \eta + \cos \varphi \zeta
\end{align*}
\]
or in matrix form
\[
\begin{bmatrix}
\vec{t} \\
\vec{n} \\
\vec{b}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \varphi & \sin \varphi \\
0 & -\sin \varphi & \cos \varphi
\end{bmatrix}
\begin{bmatrix}
\vec{\xi} \\
\vec{\eta} \\
\vec{\zeta}
\end{bmatrix}.
\]

2.5.3. Some Relations between \(a, b, c\) invariants (Curvatures of a Strip) and \(\kappa, \tau\) invariants (Curvatures of a Curve). We know that a curve \(\alpha\) has two curvatures \(\kappa\) and \(\tau\). A curve has a strip and a strip has three curvatures \(k_n, k_g\) and \(t_r\).
\[
\begin{align*}
k_n &= -b \\
k_g &= c \\
t_r &= a
\end{align*}
\]
([4],[6]). From the derivative equations we can write
\[
\xi = \eta \eta - b \zeta.
\]
If we substitute \(\vec{\xi} = \vec{t}\) in last equation, we obtain
\[
\dot{\xi} = \kappa n
\]
and
\[
\begin{align*}
b &= -\kappa \sin \varphi \\
c &= \kappa \cos \varphi
\end{align*}
\]
([4],[8]). From last two equations we obtain,
\[
\kappa^2 = b^2 + c^2.
\]
This equation is a relation between the curvature \(\kappa\) of a curve \(\alpha\) and normal curvature and geodesic curvature of a strip ([6],[10]).

By using similar operations, we obtain a new equation as follows
\[
\tau = -a + \frac{\dot{b}c - bc}{b^2 + c^2}
\]
This equation is a relation between \( \tau \) (torsion or second curvature of \( \alpha \)) and \( a, b, c \) curvatures of a strip that belongs to the curve \( \alpha \).

And also we can write

\[
a = \varphi' + \tau.
\]

**The special case:** if \( \varphi \) = constant, then \( \varphi' = 0 \). So the equation is \( a = \tau \). That is, if the angle is constant, then torsion of the strip is equal to torsion of the curve.

**Definition 2.6.** Let \( \alpha \) be a curve in \( M \subset E^3 \). If the geodesic curvature (torsion) of the curve \( \alpha \) is equal to zero, then the curve-surface pair \( (\alpha, M) \) is called a curvature strip ([6]).

### 3. General Helix

**Definition 3.1.** Let \( \alpha \) be a curve in \( E^3 \) and \( V_1 \) be the first Frenet vector field of \( \alpha.U \in \chi(E^3) \) be a constant unit vector field. If

\[
\langle V_1, U \rangle = \cos \varphi \quad \text{(constant)}
\]

\( \alpha, \varphi \) and \( \text{Sp}\{U\} \) is called an general helix, the slope angle and the slope axis ([1], [2]).

**Definition 3.2.** A regular curve is called a general helix if its first and second curvatures \( \kappa, \tau \) are not constant but \( \frac{\kappa}{\tau} \) is constant ([1], [7]).

**Definition 3.3.** A curve is called a general helix or cylindrical helix if its tangent makes a constant angle with a fixed line in space. A curve is a general helix if and only if the ratio \( \frac{\kappa}{\tau} \) is constant ([8]).

**Definition 3.4.** A helix is a curve in 3-dimensional space. The following parametrisation in Cartesian coordinates defines a helix ([12]).

\[
\begin{align*}
x(t) &= \cos t \\
y(t) &= \sin t \\
z(t) &= t.
\end{align*}
\]

As the parameter \( t \) increases, the point \( (x(t), y(t), z(t)) \) traces a right-handed helix of pitch \( 2\pi \) and radius 1 about the \( z \)-axis, in a right-handed coordinate system. In cylindrical coordinates \( (r, \theta, h) \), the same helix is parametrised by

\[
\begin{align*}
r(t) &= 1 \\
\theta(t) &= t \\
\ h(t) &= t.
\end{align*}
\]

**Definition 3.5.** If the curve \( \alpha \) is a general helix, the ratio of the first curvature of the curve to the torsion of the curve must be constant. The ratio \( \frac{\kappa}{\tau} \) is called first Harmonic Curvature of the curve and is denoted by \( H_1 \) or \( H \).

**Theorem 3.6.** A regular curve \( \alpha \subset E^3 \) is a general helix if and only if \( H(s) = \frac{k_1}{k_2} = \text{constant for } \forall s \in I \ ([2]) \).
Proof. \(( \Rightarrow )\) Let \(\alpha\) be a general helix. The slope axis of the curve \(\alpha\) is shown as \(Sp\{U\}\). Note that
\[
\langle \alpha(s), U \rangle = \cos \varphi = \text{constant}.
\]
If the Frenet trehold is \(\{V_1(s), V_2(s), V_3(s)\}\) at the point \(\alpha(s)\), then we have
\[
\langle V_1(s), U \rangle = \cos \varphi.
\]
If we take derivative of the both sides of the last equation, then we have
\[
\langle k_1(s)V_2(s), U \rangle = 0 \Rightarrow \langle V_2(s), U \rangle = 0.
\]
Hence
\[
U \in Sp\{V_1(s), V_3(s)\}.
\]
Therefore
\[
U = \cos \varphi \, V_1(s) + \sin \varphi \, V_3(s).
\]
\(U\) is the linear combination of \(V_1(s)\) and \(V_3(s)\). By differentiating the equation \(\langle V_2(s), U \rangle = 0\), we obtain
\[
\langle -k_1(s)V_1(s) + k_2(s)V_3(s), U \rangle = 0
\]
\[
-k_1(s) \langle V_1(s), U \rangle + k_2(s) \langle V_3(s), U \rangle = 0
\]
\[
-k_1(s) \cos \varphi + k_2(s) \sin \varphi = 0.
\]
By using the last equation, we see that
\[
H = \text{constant}.
\]
\((\Leftarrow)\) Let \(H(s)\) be constant for \(\forall s \in I\), and \(\lambda = \tan \varphi\), then we obtain
\[
U = \cos \varphi \, V_1(s) + \sin \varphi \, V_3(s).
\]
1) If \(U\) is a constant vector, then we have
\[
D\alpha^U = (k_1(s) \cos \varphi - \sin \varphi \, k_2(s))V_2(s).
\]
By substituting \(H(s) = \tan \varphi\) is in the last equation, we see that
\[
k_1(s) \cos \varphi - k_2(s) \sin \varphi = 0,
\]
and so
\[
U = \text{constant}.
\]
2) If \(\alpha\) is an inclined curve with slope axis \(Sp\{U\}\). Since
\[
\langle \alpha'(s), U \rangle = \langle V_1(s), \cos \varphi \, V_1(s) + \sin \varphi \, V_3(s) \rangle
\]
\[
= \cos \varphi \, \langle V_1(s), V_1(s) \rangle + \sin \varphi \, \langle V_1(s), V_3(s) \rangle
\]
we obtain
\[
\langle \alpha'(s), U \rangle = \cos \varphi = \text{constant}.
\]
**Definition 3.7.** Let $\alpha$ be a helix that lie on the cylinder. A helix which lies on the cylinder is called cylindrical helix.

![Cylindrical helix](image)

**Figure 5** Cylindrical helix

**Definition 3.8.** Let $M$ be a cylinder in $E^3$, and $\alpha$ be a helix on $M$. We define a surface element of $M$ as the part of a tangent plane at the neighbourhood of a point of the cylindrical helix. The locus of the surface element along the cylindrical helix is called a helix strip.

**Definition 3.9.** Let $M$ be a cylinder in $E^3$, and $\alpha$ be a helix on $M$. The part of the tangent plane on the cylindrical helix is called the surface element of the cylinder. The locus of the surface element along the cylindrical helix is called a strip of cylindrical helix.

**Theorem 3.10.** Suppose that $\kappa \neq 0$. Then $\alpha$ is a strip of cylindrical helix if and only if the ratio $\frac{\xi}{\tau}$ is constant.

**Proof.** Let $\theta$ be the angle between the tangent vector field $\xi$ and slope vector $u$ of a strip of cylindrical helix. Since $\xi,u = \cos \theta$ is constant, we have

$$0 = (\xi,u) = \dot{\xi} u = \kappa \xi, u$$
Because $\kappa \neq 0$ and $\zeta \cdot u = 0$, we see that $u$ is perpendicular to $\zeta$ and so
\[ u = \cos \theta \xi + \sin \theta \eta. \]
By differentiating the last equation,
\[ (\kappa \cos \theta - \tau \sin \theta) \zeta = 0 \]
or
\[ \tan \theta = \frac{\kappa}{\tau}. \]
Since $\tan \theta$ is constant, $\frac{\xi}{\tau}$ is also constant ([9]).

**Theorem 3.11.** (Terquem Theorem) Let $M_1$, $M_2$ be two different surfaces in $E^3$. Let $\alpha$ and $\beta$ be nonplanar curve in $M_1$ and a curve on $M_2$.

i. The points of the curves $\alpha$ and $\beta$ corresponds to each other 1:1 on a plane $\varepsilon$ which rolls on the $M_1$ ve $M_2$, such that the distance is constant between corresponding points.

ii. $(\alpha, M_1)$ is a curvature strip.

iii. $(\beta, M_2)$ is a curvature strip ([6]).

Claim: Two of the three lemmas gives third ([6]).

\[ \text{Figure 6} \]
By applying the similar way in the proof of Theorem II.3.11 in [6] to the strip of cylindric helix strip, we give the following theorem.

**Theorem 3.12.** Let $L$ and $M$ be a cylindrical helix and a surface in $E^3$. Suppose that $L$ and $M$ have common tangent plane along $\beta$ and cylindrical helix $\alpha$. If the curve-surface pair $(\beta, M)$ is a curvature strip, then the curve $\beta$ is a helix strip.

**Proof.**

![Figure 7 The cylinder $L$ and the surface $M$.](image)

If the curve $\alpha$ is a helix on $L$, then it provides $\frac{\xi_1}{s_1}$ is constant. We have to show that $\beta$ is a helix strip on $M$, that is, $\frac{\frac{\xi_1}{s}}{\frac{2}{s^2}}$ = constant.

By the Figure 7, we have

$$\beta(s_1) = \alpha(s_1) + \lambda(s_1) \overrightarrow{v}(s_1) \quad (2)$$

where

$$\alpha(s_1) = \overrightarrow{m} + r \xi_1(s_1). \quad (3)$$

By differentiating both side of (3), we see that

$$\frac{\overrightarrow{\xi}_1}{ds_1} = \frac{d\overrightarrow{\xi}_1}{ds_1}.$$

By (1),

$$\frac{\overrightarrow{\xi}_1}{s_1} = r(b_1 \overrightarrow{\xi}_1 - a_1 \overrightarrow{\eta}_1),$$

we obtain $a_1 = 0$ and $b_1 = 1$.

Let $r$ be the distance between gravity center of the cylinder and $\alpha(s_1)$. We denote
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If \( \vec{r} \) is a position vector of the gravity center of cylinder, then \( \vec{m} \) must be a constant vector.

Since \( a_1 = 0 \), \((\alpha, L)\) is a curvature strip. By the strips \((\alpha, L)\) and \((\beta, M)\) are curvature strips and by Theorem 17, we see that \( \lambda \) is non-zero constant. Let \( \vec{v}(s_1) \) be a vector in \( \text{Sp}\{\xi_1, \eta_1\} \), and let \( \varphi \) be the angle between \( \xi_1 \) and \( \vec{v}(s_1) \). Then we write

\[
\vec{v}(s_1) = \cos \varphi \xi_1 + \sin \varphi \eta_1.
\]

By substituting (3) and (4) in (2), and differentiating both sides, we obtain (5).

\[
\frac{d\beta}{ds_1} = \frac{d\vec{m}}{ds_1} + \frac{d\xi_1}{ds_1} + \frac{d\lambda}{ds_1} (\cos \varphi \xi_1 + \sin \varphi \eta_1) + \lambda(s_1) \frac{d(\cos \varphi \xi_1 + \sin \varphi \eta_1)}{ds_1}.
\]

Since the vector \( \vec{m} \) and \( \lambda \) are constant, we obtain the following equation

\[
\frac{d\beta}{ds_1} = \frac{d\xi_1}{ds_1} + \lambda(s_1) \frac{d(\cos \varphi \xi_1 + \sin \varphi \eta_1)}{ds_1}
\]

or

\[
\frac{d\beta}{ds_1} = \frac{d\xi_1}{ds_1} + \lambda(s_1) (-\frac{d\varphi}{ds_1} \sin \varphi \xi_1 + \cos \varphi \frac{d\xi_1}{ds_1}) + \frac{d\varphi}{ds_1} \cos \varphi \eta_1 + \sin \varphi \frac{d\eta_1}{ds_1}.
\]

By (1), we see that

\[
\frac{d\beta}{ds_1} = \left[ 1 - \lambda(\frac{d\varphi}{ds_1} + c_1) \sin \varphi \right] \xi_1 + \lambda(\frac{d\varphi}{ds_1} + c_1) \cos \varphi \eta_1 - \lambda \cos \varphi \xi_1.
\]

Since the cylindric helix and the surface \( M \) have the same tangent plane along the curves \( \alpha \) and \( \beta \), we can write

\[
\left( \frac{d\beta}{ds_1}, \xi_1 \right) = 0.
\]

By substituting (6) in the last equation, we obtain \( \cos \varphi = 0 \). By using that equation in (6), we have

\[
\frac{d\beta}{ds_1} = (1 + \lambda c_1) \xi_1
\]

If we calculate the second and third derivatives of the curve \( \beta \), then we get

\[
\frac{d^2\beta}{ds_1^2} = -\lambda c_1 \xi_1 + (1 + \lambda c_1) \eta_1 - (1 + \lambda c_1) \xi_1
\]

\[
\frac{d^3\beta}{ds_1^3} = \left[ 1 + \lambda c_1 \xi_1 - (1 + \lambda c_1) c_1^2 - (1 + \lambda c_1) \right] \xi_1 + \left[ -\lambda c_1 \xi_1 \right] \eta_1 + \left[ -\lambda c_1 \eta_1 \right] \xi_1 + \left[ \lambda c_1 \xi_1 \right] \eta_1.
\]

Since the same result is obtained by using the other form of (7), we use the form

\[
\frac{d^2\beta}{ds_1^2} = (1 - \lambda c_1) \xi_1
\]

of (7) in the rest of our proof. By differentiating both sides of
(7), we obtain
\[
\frac{d\beta}{ds_1} = (1 - \lambda c_1) \xi_1
\]
\[
\frac{d^2\beta}{ds_1^2} = -\lambda c_1 \xi_1 + (1 - \lambda c_1) c_1 \eta_1 - (1 - \lambda c_1) \xi_1
\]
\[
\frac{d^3\beta}{ds_1^3} = \left[ -\lambda c_1' - (1 - \lambda c_1) c_1^2 - (1 - \lambda c_1) \right] \xi_1 + \left[ -3\lambda c_1 c_1' + c_1' \right] \eta_1 + 2\lambda c_1 \xi_1.
\]

By applying Gram-Schmidt to the \(\{\beta, \beta', \beta''\}\), then we have
\[
F_1 = (1 - \lambda c_1) \xi_1
\]
\[
F_2 = (1 - \lambda c_1) c_1 \eta_1 - (1 - \lambda c_1) \xi_1
\]
\[
F_3 = \frac{(1 - \lambda c_1) c_1'}{c_1^2 + 1} \eta_1 + \frac{(1 - \lambda c_1) c_1' c_1}{c_1^2 + 1} \xi_1.
\]

By [6], we have
\[
\kappa_1^2 = b_1^2 + c_1^2, \quad b_1 = 1 \tag{8}
\]
and
\[
\tau_1^2 = -a_1 + \frac{b_1' c_1 - b_1 c_1'}{b_1^2 + c_1^2}, \quad a_1 = 0 \tag{9}
\]

By (8) and (9), we see that
\[
\tau_1 = \frac{-c_1'}{\kappa_1^2} \tag{10}
\]

By using (10) in \(F_3\), we obtain
\[
F_3 = -(1 - \lambda c_1) \tau_1 \eta_1 - (1 - \lambda c_1) c_1 \tau_1 \xi_1.
\]

If we calculate \(\kappa_2\) and \(\tau_2\), then we have
\[
\kappa_2 = \frac{\kappa_1}{|1 - \lambda c_1|}
\]
and
\[
\tau_2 = \frac{\tau_1}{|1 - \lambda c_1|}
\]

Dividing by \(\kappa_2\) to \(\tau_2\), we obtain
\[
\frac{\kappa_2}{\tau_2} = \frac{\kappa_1}{\tau_1}. \tag{11}
\]

We obtain the proof of theorem from last equation. \(\square\)

ÖZET: Bu çalışmada silindirik helis şeridleri incelendi. Yeni bir tanım ve bir karekterizasyon verildi. Genel helis ve Terquem teoremlerinin (herhangi iki yüzey arasındaki uzaklık sabit olmasıyla ilgili Joachimsthal teoremlerinden biri) karekterizasyonlarından yararlanıldı.
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