UNIT DUAL QUATERNIONS AND ARCS

HESNA KABADAYI AND YUSUF YAYLI

ABSTRACT. In this paper we obtain sine and cosine rules for dual spherical triangle on the dual unit sphere $\tilde{S}^2$ by representing great circle arcs by dual quaternions.

1. Introduction

As it is well known that an arc of a unit circle (subtending an angle $\theta$ at the origin) can be represented by a complex number of unit norm $\cos \theta + i \sin \theta$.

Great circle arcs on a unit sphere represented by a unit quaternion and sine and cosine rules are obtained by J. P. Ward (see [1], pg. 98-102).

A similar correspondence is possible with dual quaternions and great circle arcs on the dual unit sphere $\tilde{S}^2$.

The sine and cosine rules for dual and real spherical trigonometry have been well known for a long time. (see [2], [3], [4], [5]).

Here in this paper we obtain sine and cosine laws by means of this correspondence between great circle arcs on dual unit sphere and dual quaternions.

2. Dual Quaternions and Arcs

Consider a unit dual quaternion $q = \cos \varphi + \hat{q} \sin \varphi$. We may associate this dual quaternion by the great circle arcs which is obtained when the diametral plane with normal $\hat{q}$ intersects the unit dual sphere.

Let $A, B$ and $C$ be the unit dual vectors and $q = (A, B) + A \wedge B$ and $p = (B, C) + B \wedge C$ be dual unit quaternions.

The quaternion product of $p$ and $q$ is readily checked to be
\[ pq = \langle A, B \rangle \langle B, C \rangle - \langle A \wedge B, B \wedge C \rangle + \langle B, C \rangle A \wedge B + \langle A, B \rangle B \wedge C + (B \wedge C) \wedge (A \wedge B) \]
\[ = \langle A, C \rangle + A \wedge C. \]

Considering that \( p \) and \( q \) are unit dual quaternions, they are screw operators at the same time. Hence \( q(A) = B \), \( p(B) = C \) implies that \( pq(A) = C \). This means that the line \( d_1 \) which corresponds \( A \) transforms into the line \( d_2 \) which corresponds \( C \).

We write (using \( \sim \) to specify the geometrical correspondence)
\[ arcAB \sim q = \langle A, B \rangle + A \wedge B \]
\[ = \cos \varphi + \hat{q} \sin \varphi \]
\[ = \cos (\theta + \varepsilon \theta^*) + \sin (\theta + \varepsilon \theta^*) \frac{A \wedge B}{\|A \wedge B\|} \]
\[ arcBC \sim p = \langle B, C \rangle + B \wedge C \]
\[ = \cos \phi + \hat{p} \sin \phi \]
\[ = \cos \phi + \sin \phi \frac{B \wedge C}{\|B \wedge C\|} \]
\[ arcAC \sim pq = \langle A, C \rangle + A \wedge C. \]

Hence we write
\[ arcAB + arcBC = arcAC \]
or
\[ arcq + arcp = arcpq. \]

**Theorem 2.1.** Let \( A_1, A_2, \ldots, A_n \) be unit dual vectors. Then
\[ arcA_1A_2 + arcA_2A_3 + \ldots + arcA_{n-1}A_n = arcA_1A_n. \]

**Proof.** Denoting \( q_kq_{k+1} \) by \( q_{k(k+1)} \) we have
\[ arcA_1A_2 \sim q_{12} = \langle A_1, A_2 \rangle + A_1 \wedge A_2 \]
\[ arcA_2A_3 \sim q_{23} = \langle A_2, A_3 \rangle + A_2 \wedge A_3 \]
\[ \vdots \]
\[ arcA_{n-1}A_n \sim q_{(n-1)n} = \langle A_{n-1}, A_n \rangle + A_{n-1} \wedge A_n \]
\[ arcA_1A_n \sim q_{1n} = \langle A_1, A_n \rangle + A_1 \wedge A_n. \]

Noting that \( q_{12}(A_1) = A_2 \) we have
\[ (q_{(n-1)n} \ldots q_{23}q_{12})(A_1) = (q_{(n-1)n} \ldots q_{23})(A_2) = q_{(n-1)n}(A_{n-1}) = A_n. \]
Hence we get
\[ q_{(n-1)n} q_{23} q_{12} = (A_1, A_n) + A_1 \wedge A_n \]
\[ = q_{1n}. \]

Thus
\[ \text{arc}(q_{(n-1)n} q_{23} q_{12}) = \text{arc}q_{1n}. \]

Therefore
\[ \text{arc}A_1 A_2 + \text{arc}A_2 A_3 + \ldots + \text{arc}A_{n-1} A_n = \text{arc}A_1 A_n \]
or
\[ \text{arc}q_{(n-1)n} + \ldots + \text{arc}q_{23} + \text{arc}q_{12} = \text{arc}q_{1n}. \]

Note that when dual quaternions are taken as real quaternions this result reduces the case in [1].

\[ \square \]

3. **The Sine and Cosine Laws for a Dual Spherical Triangle**

We consider two different points \( A \) and \( B \) on the dual unit sphere given by dual unit vectors \( \hat{x} = x + \varepsilon x^* \), \( \hat{y} = y + \varepsilon y^* \) respectively. We introduce the set of all dual vectors given by \( \hat{c}_\lambda = c_\lambda + \varepsilon c^*_\lambda = (1 - \hat{\lambda}) \hat{\lambda} + \hat{\lambda} \hat{y}_\lambda \), where \( \hat{\lambda} = \lambda + \varepsilon \lambda^* \) and \( 0 \leq \lambda \leq 1 \). We put \( \hat{c}_\lambda = |c_\lambda| \hat{e}_\lambda \); then \( \hat{e}_\lambda \) is a point \( C_\lambda \) on the dual unit sphere. The set of all points \( C_\lambda \) with \( 0 \leq \lambda \leq 1 \) is called the dual great-circle arc \( \text{arc}AB \). We will say that \( C_\lambda \) runs along \( \text{arc}AB \) from \( A \) to \( B \) if \( \lambda \) increases from 0 to 1. With the \( \text{arc}AB \) we will always mean this arc in the sense from \( A \) to \( B \).

Let \( A, B \) and \( C \) be three points on the dual unit sphere \( S^2 \) given by the linearly independent dual unit vectors \( \hat{x} = x + \varepsilon x^* \), \( \hat{y} = y + \varepsilon y^* \) and \( \hat{z} = z + \varepsilon z^* \) respectively. We will always suppose that the notations is such that \( \det(x, y, z) > 0 \). These points together with the dual great-circle arcs \( \text{arc}AB, \text{arc}BC, \text{arc}CA \) form a dual spherical triangle \( \Delta ABC \). (see [2])

Having defined a dual spherical triangle there is naturally defined six dual angles \( a^o = a + \varepsilon a^* \), \( b^o = b + \varepsilon b^* \), \( c^o = c + \varepsilon c^* \) called arc angles and \( A^o = u + \varepsilon u^* \), \( B^o = v + \varepsilon v^* \), \( C^o = w + \varepsilon w^* \) called vertex angles (see figure 3.1).
We can represent arcs dual quaternionically. If \( q = \cos a^o + \hat{q} \sin a^o \),
\[ p = \cos c^o + \hat{p} \sin c^o \]
then
\[ qp = \cos c^o \cos a^o - \langle \hat{q}, \hat{p} \rangle \sin c^o \sin a^o + \hat{q} \sin a^o \cos c^o + \hat{p} \cos a^o \sin c^o + \hat{q} \hat{p} \sin c^o \sin a^o. \]
On the other hand \( \text{arc} AB \sim p, \text{arc} BC \sim q, \text{arc} AC \sim qp \) and writing \( \text{arc} AC \sim \cos b^o + \hat{m} \sin b^o \) we get, by equating scalar and vector parts:
\[ \cos c^o \cos a^o - \langle \hat{q}, \hat{p} \rangle \sin c^o \sin a^o = \cos b^o \]
(3.1)
\[ \hat{q} \sin a^o \cos c^o + \hat{p} \cos a^o \sin c^o + \hat{q} \hat{p} \sin c^o \sin a^o = \hat{m} \sin b^o. \]
(3.2)
Note that \( \hat{p}, \hat{q}, \hat{m} \) are unit dual vectors in the direction of \( A \hat{B}, B \hat{C} \) and \( A \hat{C} \) respectively i.e.
\[ \hat{p} = \frac{A \wedge B}{\|A \wedge B\|}, \hat{q} = \frac{B \wedge C}{\|B \wedge C\|}, \hat{m} = \frac{A \wedge C}{\|A \wedge C\|}. \]
We define
\[ \langle B, C \rangle = \cos a^o, \ B \wedge C = \hat{q} \sin a^o. \]
Similar definitions are given for dual angle \( b^o \) and \( c^o \).

If \( a^o = a + \varepsilon a^* \), we have consequently \( \sin a > 0 \). This implies (see [2]) \( |\sin a^o| = \sin a^o \). Similarly \( |\sin b^o| = \sin b^o, |\sin c^o| = \sin c^o \). It is moreover readily seen that \( A, B, C \) are dual unit vectors having the same sense as \( \hat{p} \wedge \hat{m}, \hat{p} \wedge \hat{q} \) and \( \hat{m} \wedge \hat{q} \) respectively. The angle \( A^o \) of \( \Delta ABC \) is defined as the dual angle given by
\[ \langle \hat{p}, \hat{m} \rangle = -\cos A^o, \ \hat{p} \wedge \hat{m} = A \sin A^o. \]
Similar definitions for the angles $B^o$ and $C^o$ are given, i.e.
\[ \langle \hat{p}, \hat{q} \rangle = -\cos B^o, \quad \hat{p} \land \hat{q} = B \sin B^o \] and \[ \langle \hat{m}, \hat{q} \rangle = -\cos C^o, \quad \hat{m} \land \hat{q} = C \sin C^o. \]
Now (3.1) implies the law of cosine in dual spherical trigonometry as follows:

**Theorem 3.1.** Let $\Delta ABC$ be a dual spherical triangle on the dual unit sphere $S^2$.
Then
\[ \cos c^o \cos a^o + \cos B^o \sin c^o \sin a^o = \cos b^o \] (3.3)
and
\[ \cos a^o = \cos b^o \cos c^o + \cos A^o \sin b^o \sin c^o \] (3.4)
\[ \cos c^o = \cos a^o \cos b^o + \cos C^o \sin a^o \sin b^o. \] (3.5)

**Corollary 1.** The real and dual parts of the formula (3.3), (3.4), (3.5) are given by
\[ \cos u = \frac{\cos a - \cos c \cos b}{\sin c \sin b}, \quad \sin u = \frac{-\sin a}{\sin b} \left( b^* \cos w + c^* \cos v - a^* \right) \]
\[ \cos v = \frac{\cos b - \cos c \cos a}{\sin c \sin a}, \quad \sin v = \frac{-\sin b}{\sin a} \left( a^* \cos w + c^* \cos u - b^* \right) \]
\[ \cos w = \frac{\cos c - \cos a \cos b}{\sin a \sin b}, \quad \sin w = \frac{-\sin c}{\sin a \sin b} \left( b^* \cos u + a^* \cos v - c^* \right). \]
Since $\hat{q} \land \hat{p} = -B \sin B^o$ and since $\langle B, \hat{p} \rangle = 0$ and $\langle B, \hat{q} \rangle = 0$, from (3.2) we get
\[ \sin B^o \sin c^o \sin a^o = -\langle B, \hat{m} \rangle \sin b^o. \]
Hence
\[ \frac{\sin B^o}{\sin b^o} = \frac{\langle B, \hat{m} \rangle}{\sin c^o \sin a^o} = -\frac{\langle B, A \land C \rangle}{\sin a^o \sin b^o \sin c^o} = \frac{\langle A, B \land C \rangle}{\sin a^o \sin b^o \sin c^o}. \]
Thus, since the right hand side is unchanged on cyclic interchange we obtain:

**Theorem 3.2.** Let $\Delta ABC$ be a dual spherical triangle on the dual unit sphere $S^2$ then
\[ \frac{\sin A^o}{\sin a^o} = \frac{\sin B^o}{\sin b^o} = \frac{\sin C^o}{\sin c^o}. \] (3.6)

**Corollary 2.** The real and dual part of the Formula (3.6) is given by
\[ \frac{\sin u}{\sin a} = \frac{\sin v}{\sin b} = \frac{\sin w}{\sin c} \]
and
\[ u^* \cos u \sin a - a^* \cot a \frac{\sin u}{\sin a} = v^* \cos v \sin b - b^* \cot b \frac{\sin v}{\sin b} = w^* \cos w \sin c - c^* \cot c \frac{\sin w}{\sin c}. \]
Note also that sine law is obtained from (3.2) by taking the scalar product of both sides with $B$. One other possibility is taking the vector product of this equation with $B$ implies

$$0 = B \wedge \hat{q} \sin a^o \cos c^o + B \wedge \hat{p} \cos a^o \sin c^o - B \wedge \hat{m} \sin b^o. \quad (3.7)$$

Note that

$$B \wedge \hat{q} = \frac{B \wedge (B \wedge C)}{\sin a^o} = \frac{[B, C] B - C}{\sin a^o}$$

and

$$B \wedge \hat{p} = \frac{B \wedge (A \wedge B)}{\sin c^o} = \frac{[A - \langle B, A \rangle B]}{\sin c^o}$$

Using $\langle B, A \rangle = \cos c^o$, $\langle B, C \rangle = \cos a^o$ and $\langle A, C \rangle = \cos b^o$ implies the identity in (3.7).

**Remark 3.3.** The results above coincide with the ones for real spherical triangles when the vectors are real.

**ÖZET:** Bu çalışmada, birim dual kuaterniyonlara dual yaylar karşılık getirilmiş, birim dual kuaterniyonları kullanarak dual küresel üçgenler için bilinen kosinus ve sinüs bağıntılar elde edilmiştir.

**REFERENCES**


*Current address:* Ankara University, Faculty of Sciences, Dept. of Mathematics, Ankara, TURKEY

*E-mail address:* kabadayi@science.ankara.edu.tr, yayli@science.ankara.edu.tr

*URL:* http://communications.science.ankara.edu.tr