ON CR–SUBMANIFOLDS OF A S–MANIFOLD ENDOVED WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. In this paper, we study CR–submanifolds of an S–manifold endowed with a semi-symmetric non-metric connection. We give an example, investigating integrabilities of horizontal and vertical distributions of CR–submanifolds endowed with a semi-symmetric non-metric connection. We also consider parallel horizontal distributions of CR–submanifolds.

1. Introduction

In 1963, Yano [23] introduced the notion of f-structure on a C∞ m-dimensional manifold M, as a non-vanishing tensor field f of type (1, 1) on M which satisfies f3 + f = 0 and has constant rank r. It is known that r is even, say r = 2n. Moreover, TM splits into two complementary subbundles Imf and ker f and the restriction of f to Imf determines a complex structure on such subbundle. It is also known that the existence of an f-structure on M is equivalent to a reduction of the structure group to U(n) × O(s) (see [9]), where s = m − 2n. In 1970, Goldberg and Yano [12] introduced globally frame f-manifolds (also called metric f- manifolds and f.pk-manifolds). A wide class of globally frame f-manifolds was introduced in [9] by Blair according to the following definition: a metric f-structure is said to be a K-structure if the fundamental 2-form Φ, defined usually as Φ(X, Y) = g(X, fY), for any vector fields X and Y on M, is closed and the normality condition holds, that is, [f, f] + 2 ∑i=1 dτi ⊗ ξi = 0, where [f, f] denotes the Nijenhuis torsion of f. A K-manifold is called an S-manifold if dτk = Φ, for all k = 1, ..., s. The S-manifolds have been studied by several authors (see, for instance, [2], [3], [5], [10], [11]).

On the other hand, the notion of a CR–submanifold of Kaeheirian manifolds was introduced by A. Bejancu in [6]. Later, the concept of CR–submanifolds has been developed by [4], [8], [13], [14], [16], [18], [19], [20], [22] and others.
Let \( \nabla \) be a linear connection in an \( n \)-dimensional differentiable manifold \( M \).

The torsion tensor \( T \) and the curvature tensor \( R \) of \( \nabla \) are given respectively by [7]

\[
T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],
\]

\[
R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.
\]

The connection \( \nabla \) is symmetric if the torsion tensor \( T \) vanishes, otherwise it is non-symmetric. The connection \( \nabla \) is a metric connection if there is a Riemannian metric \( g \) in \( M \) such that \( \nabla g = 0 \), otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection.

In [17], Friedmann and Schouten introduced the notion of semi-symmetric linear connections. More precisely, if \( \nabla \) is a linear connection in a differentiable manifold \( M \), the torsion tensor \( T \) of \( \nabla \) is given by

\[
T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],
\]

for any vector fields \( X \) and \( Y \) on \( M \). The connection \( \nabla \) is said to be semi-symmetric if its torsion tensor \( T \) vanishes, otherwise it is said to be non-symmetric. In this case, \( \nabla \) is said to be a semi-symmetric connection if its torsion tensor \( T \) is of the form

\[
T(X,Y) = \eta(Y)X - \eta(X)Y,
\]

for any \( X, Y \), where \( \eta \) is a 1-form on \( M \). Moreover, if \( g \) is a (pseudo)-Riemannian metric on \( M \), \( \nabla \) is called a metric connection if \( \nabla g = 0 \), otherwise it is called non-metric. It is well known that the Riemannian connection is the unique metric and symmetric linear connection on a Riemannian manifold. In 1932, Hayden [15] defined a metric connection with torsion on a Riemannian manifold. In [1] Agashe and Chafee defined a semi-symmetric non-metric connection on a Riemannian manifold and studied some of its properties.

Later, the concept of semi-symmetric non-metric connection has been developed by (see, for instance, [3], [21]) and others. In this paper we study \( CR \)-submanifolds of an \( S \)-manifold endowed with a semi-symmetric non-metric connection. We consider integra
tibilities of horizontal and vertical distributions of \( CR \)-submanifolds with a semi-symmetric non-metric connection. We also consider parallel horizontal distributions of \( CR \)-submanifolds.

The paper is organized as follows: In section 2, we give a brief introduction to \( S \)-manifolds. In section 3, we study \( CR \)-submanifolds of \( S \)-manifolds. We find necessary conditions for the induced connection on a \( CR \)-submanifold of an \( S \)-manifold with semi-symmetric non-metric connection to be also a semi-symmetric non-metric connection. In section 4, We study integrabilities of horizontal and vertical distributions of \( CR \)-submanifolds with a semi-symmetric non-metric connection.

2. \( S \)-Manifolds

A \((2n+s)\)-dimensional differentiable manifold \( \tilde{M} \) is called a metric \( f \)-manifold if there exist an \((1, 1)\) type tensor field \( f \), \( s \) vector fields \( \xi_1, \ldots, \xi_s \), \( s \) 1-forms \( \eta_1, \ldots, \eta_s \)
and a Riemannian metric $g$ on $\widetilde{M}$ such that

$$f^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i, \quad \eta^i(\xi_j) = \delta_{ij}, \quad f\xi_i = 0, \quad \eta^i \circ f = 0, \quad (2.1)$$

$$g(fX, fY) = g(X, Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y), \quad (2.2)$$

for any $X, Y \in \Gamma(T\widetilde{M})$, $i, j \in \{1, \ldots, s\}$. In addition we have:

$$\eta^i(X) = g(X, \xi_i), \quad g(X, fY) = -g(fX, Y). \quad (2.3)$$

Moreover, a metric $f$-manifold is normal if

$$[f, f] + 2\sum_{\alpha=1}^{s} d\eta^\alpha \otimes \xi_\alpha = 0$$

where $[f, f]$ is Nijenhuis tensor of $f$.

Then a 2-form $F$ is defined by $F(X, Y) = g(X, fY)$, for any $X, Y \in \Gamma(T\widetilde{M})$, called the fundamental 2-form. Then $\widetilde{M}$ is said to be an $S$-manifold if the $f$ structure is normal and

$$\eta^1 \wedge \ldots \wedge \eta^s \wedge (d\eta^\alpha)^n \neq 0, \quad F = d\eta^\alpha$$

for any $\alpha = 1, \ldots, s$. In the case $s = 1$, an $S$–manifold is a Sasakian manifold.

Now, if $\nabla$ denotes the Riemannian connection associated with $g$, then $[7]$

$$\left(\nabla_X f\right) Y = \sum_{\alpha=1}^{s} \left\{ g(fX, fY) \xi_\alpha + \eta^\alpha(X) f^2 X \right\}, \quad (2.4)$$

for all $X, Y \in \Gamma(T\widetilde{M})$. From (2.4), it is deduced that

$$\nabla_X \xi_\alpha = -fX, \quad (2.5)$$

for any $X, Y \in \Gamma(T\widetilde{M})$, $\alpha \in \{1, \ldots, s\}$.

### 3. CR–Submanifold of S–Manifolds

**Definition 3.1.** An $(2m+s)$–dimensional Riemannian submanifold $M$ of $S$–manifold $\widetilde{M}$ is called a CR–submanifold if $\xi_1, \xi_2, \ldots, \xi_s$ is tangent to $M$ and there exists on $M$ two differentiable distributions $D$ and $D^\perp$ on $M$ satisfying:

(i) $TM = D \oplus D^\perp \oplus sp\{\xi_1, \ldots, \xi_s\}$;

(ii) The distribution $D$ is invariant under $f$, that is $fD_x = D_x$ for any $x \in M$;

(iii) The distribution $D^\perp$ is anti-invariant under $f$, that is, $fD^\perp_x \subseteq T^\perp_x M$ for any $x \in M$, where $T_x M$ and $T^\perp_x M$ are the tangent space of $M$ at $x$.

We denote by $2p$ and $q$ the real dimensions of $D_x$ and $D^\perp_x$ respectively, for any $x \in M$. Then if $p = 0$ we have an anti-invariant submanifold tangent to $\xi_1, \xi_2, \ldots, \xi_s$ and if $q = 0$, we have an invariant submanifold.
Example 3.1. In what follows, \((R^{2n+s}, f, \eta, \xi, g)\) will denote the manifold \(R^{2n+s}\) with its usual \(S\)-structure given by

\[
\eta^\alpha = \frac{1}{2}(dz_\alpha - \sum_{i=1}^{n} y_i dx_i), \quad \xi_\alpha = 2\frac{\partial}{\partial z_\alpha}
\]

\[
f(\sum_{i=1}^{n} (X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}) + \sum_{\alpha=1}^{s} Z_\alpha \frac{\partial}{\partial z_\alpha}) = \sum_{i=1}^{n} (Y_i \frac{\partial}{\partial x_i} - X_i \frac{\partial}{\partial y_i}) + \sum_{\alpha=1}^{s} \sum_{i=1}^{n} Y_i y_i \frac{\partial}{\partial z_\alpha}
\]

\[
g = \sum_{\alpha=1}^{s} \eta^\alpha \otimes \eta^\alpha + \frac{1}{4} \sum_{i=1}^{n} (dx_i \otimes dx_i + dy_i \otimes dy_i),
\]

\((x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_s)\) denoting the Cartesian coordinates on \(R^{2n+s}\). The consider a submanifold of \(R^8\) defined by

\[
M = X(u, v, k, l, t_1, t_2) = 2(u, 0, k, v, l, 0, t_1, t_2).
\]

Then local frame of \(TM\)

\[
e_1 = 2\frac{\partial}{\partial x_1}, \quad e_2 = 2\frac{\partial}{\partial y_1},
\]

\[
e_3 = 2\frac{\partial}{\partial x_3}, \quad e_4 = 2\frac{\partial}{\partial y_2},
\]

\[
e_5 = 2\frac{\partial}{\partial z_1} = \xi_1, \quad e_6 = 2\frac{\partial}{\partial z_2} = \xi_2
\]

and

\[
e_1^* = 2\frac{\partial}{\partial x_2}, \quad e_2^* = 2\frac{\partial}{\partial y_3}
\]

from a basis of \(T^\perp M\). We determine \(D_1 = \text{sp}\{e_1, e_2\}\) and \(D_2 = \text{sp}\{e_3, e_4\}\), then \(D_1, D_2\) are invariant and anti-invariant distribution. Thus \(TM = D_1 \oplus D_2 \oplus \text{sp}\{\xi_1, \xi_2\}\) is a \(CR\)-submanifold of \(R^8\).

Let \(\tilde{\nabla}\) be the Levi-Civita connection of \(\tilde{M}\) with respect to the induced metric \(g\). Then Gauss and Weingarten formulas are given by

\[
\tilde{\nabla}_X Y = \nabla^*_X Y + h(X, Y)
\]

(3.1)

and

\[
\tilde{\nabla}_X N = -A_N X + \nabla^*_X N
\]

(3.2)

for any \(X, Y \in \Gamma(TM)\) and \(N \in \Gamma(T^\perp M)\). \(\nabla^*_X\) is the connection in the normal bundle, \(h\) is the second fundamental from of \(\tilde{M}\) and \(A_N\) is the Weingarten endomorphism associated with \(N\). The second fundamental form \(h\) and the shape operator \(A\) related by

\[
g(h(X, Y), N) = g(A_N X, Y)
\]

(3.3)

for any \(X, Y \in \Gamma(TM)\) and \(N \in \Gamma(T^\perp M)\).
A CR–submanifold is said to be \( D \)–totally geodesic if \( h(X, Y) = 0 \) for any \( X, Y \in \Gamma(D) \) and it is said to be \( D^\perp \)–totally geodesic if \( h(Z, W) = 0 \) for any \( Z, W \in \Gamma(D^\perp) \).

The projection morphisms of \( TM \) to \( D \) and \( D^\perp \) are denoted by \( P \) and \( Q \) respectively. For any \( X, Y \in \Gamma(TM) \) and \( N \in \Gamma(T^\perp M) \) we have

\[
X = PX + QX + \sum_{\alpha=1}^{s} \eta^\alpha(X) \xi_\alpha, \quad 1 \leq \alpha \leq s \quad (3.4)
\]

\[
fN = BN + CN \quad (3.5)
\]

where \( BN \) (resp. \( CN \)) denotes the tangential (resp. normal) component of \( \varphi N \).

Now, we define a connection \( \nabla \) as

\[
\nabla_X Y = \tilde{\nabla}_X Y + \sum_{\alpha=1}^{s} \eta^\alpha(Y) X.
\]

**Theorem 3.1.** Let \( \tilde{\nabla} \) be the Riemannian connection on a \( S \)–manifold \( \tilde{M} \). Then the linear connection which is defined as

\[
\nabla_X Y = \tilde{\nabla}_X Y + \sum_{\alpha=1}^{s} \eta^\alpha(Y) X, \quad \forall X, Y \in \Gamma(TM) \quad (3.6)
\]

is a semi-symmetric non metric connection on \( \tilde{M} \).

**Proof.** Using new connection and the fact that the Riemannian connection is torsion free, the torsion tensor \( T \) of the connection \( \nabla \) is given by

\[
T(X, Y) = \sum_{\alpha=1}^{s} \{\eta^\alpha(Y)X - \eta^\alpha(X)Y\} \quad (3.7)
\]

for all \( X, Y \in \Gamma(TM) \). Moreover, by using (3.6) again, for all \( X, Y, Z \in \Gamma(TM) \) and since \( \tilde{\nabla} \) is a metric connection, we have

\[
(\tilde{\nabla}_X g)(Y, Z) = -\sum_{\alpha=1}^{s} \{g(X, Y)\eta^\alpha(Z) - g(X, Z)\eta^\alpha(Y)\}. \quad (3.8)
\]

From (3.7) and (3.8) we conclude that the linear connection \( \tilde{\nabla} \) is a semi-symmetric non-metric connection on \( \tilde{M} \). \( \Box \)

**Theorem 3.2.** Let \( M \) be a CR submanifold of \( S \)–manifold \( \tilde{M} \). Then

\[
(\tilde{\nabla}_X f)Y = \sum_{\alpha=1}^{s} \{g(X, Y)\xi_\alpha - \eta^\alpha(Y)(X + fX)\} \quad (3.9)
\]

for all \( X, Y \in \Gamma(TM) \).
Proof. From (3.6), we get
\[
(\nabla_X f)Y = \sum_{\alpha=1}^{s} \{g(X, Y)\xi_\alpha - \eta^\alpha (Y) X - \eta^\alpha (Y) f X\}
\]
for all \(X, Y \in \Gamma(TM)\). Therefore we obtain the result from (2.4).

\[\square\]

**Corollary 3.1.** Let \(M\) be a CR submanifold of S-manifold \(\widetilde{M}\) with semi-symmetric non-metric connection \(\nabla\). Then
\[
\nabla_X \xi_\alpha = -fX + X
\]
for all \(X \in \Gamma(TM)\).

We denote by same symbol \(g\) both metrics on \(\widetilde{M}\) and \(M\). Let \(\nabla\) be the semi-symmetric non-metric connection on \(\widetilde{M}\) and \(\nabla\) be the induced connection on \(M\). Then,
\[
\nabla_X Y = \nabla_X^* Y + m(X, Y)
\]
where \(m\) is a \(\Gamma(T^\perp M)\)–valued symmetric tensor field on CR-submanifold \(M\). If \(\nabla^*\) denotes the induced connection from the Riemannian connection \(\nabla\), then
\[
\nabla_X Y = \nabla_X^* Y + h(X, Y),
\]
where \(h\) is the second fundamental form. Using (3.1) and (3.4), we have
\[
\nabla_X Y + m(X, Y) = \nabla_X^* Y + h(X, Y) + \sum_{\alpha=1}^{s} \eta^\alpha (Y) X.
\]
Equating tangential and normal components from both the sides, we get
\[
m(X, Y) = h(X, Y)
\]
and
\[
\nabla_X Y = \nabla_X^* Y + \sum_{i=1}^{s} \eta^\alpha(Y) X.
\]
Thus \(\nabla\) is also a semi-symmetric non-metric connection. From (3.2) and (3.13), we have
\[
\nabla_X N = \nabla_X^* N + \sum_{\alpha=1}^{s} \eta^\alpha(N) X
\]
\[
= -A_N X + \nabla_X^\perp N + \sum_{\alpha=1}^{s} \eta^\alpha(N) X,
\]
where \(X \in \Gamma(TM)\) and \(N \in \Gamma(T^\perp M)\).

Now, Gauss and Weingarten formulas for a CR-submanifolds of a S-manifold with a semi-symmetric non-metric connection is
\[
\nabla_X Y = \nabla_X Y + h(X, Y)
\]
\[
\n\nabla_X N = -A_N X + \nabla_X^\perp N + \sum_{\alpha=1}^{s} \eta^\alpha(N) X
\]

(3.15)

for any \( X, Y \in \Gamma(TM) \) and \( N \in \Gamma(T^\perp M) \), \( h \) second fundamental form of \( M \) and \( A_N \) is the Weingarten endomorphism associated with \( N \).

**Theorem 3.3.** The connection induced on \( CR \)-submanifold of a \( S \)-manifold with semi-symmetric non-metric connection is also a semi-symmetric non-metric connection.

**Proof.** From (3.7) and (3.8), we get

\[
\tilde{T}(X,Y) = T(X,Y)
\]

and

\[
(\nabla_X g)(Y,Z) = (\nabla_X g)(Y,Z)
\]

for any \( X, Y \in \Gamma(TM) \), where \( T \) is the torsion tensor of \( \nabla \). \( \Box \)

**4. Integrability and Parallel of Distributions**

**Lemma 4.1.** Let \( M \) be a \( CR \)-submanifold of an \( S \)-manifold \( \tilde{M} \) with semi-symmetric non-metric connection. Then,

\[
P\nabla_X fPY - PA_{fQY}X - fP\nabla_X Y = -\sum_{\alpha=1}^{s} \eta^\alpha(Y)(PX + fPX), \quad (4.1)
\]

\[
Q\nabla_X fPY - QA_{fQY}X - th(X,Y) = -\sum_{\alpha=1}^{s} \eta^\alpha(Y)QX, \quad (4.2)
\]

\[
h(X,fPY) - fQ\nabla_X Y + \nabla_X^\perp fQY = nh(X,Y) - \sum_{\alpha=1}^{s} \eta^\alpha(Y)fQX, \quad (4.3)
\]

for all \( X, Y \in \Gamma(TM) \).

**Proof.** By direct covariant differentiation, we have

\[
\nabla_X fY = (\nabla_X f)Y + f(\nabla_X Y).
\]

for any \( X, Y \in \Gamma(TM) \). By virtue of (3.4),(3.9),(3.14) and (3.15) we get

\[
\nabla_X fPY + h(X,fPY) + \left(-A_{fQY}X + \nabla_X^\perp fQY\right)
\]

\[
= \sum_{\alpha=1}^{s} \left\{ g(X,Y) \xi_\alpha - \eta^\alpha(Y)(fX + X)\right\} + f\nabla_X Y + fh(X,Y).
\]
Using (3.4) again, we have
\[
P \nabla_X fP Y + Q \nabla_X fP Y + h (X, fP Y) - PA_{fQ Y} X - QA_{fQ Y} X + \nabla_X^f Q Y \]
\[
= \sum_{\alpha=1}^{s} \{ g (X, Y) P \xi_\alpha + g (X, Y) Q \xi_\alpha - \eta^\alpha (Y) P X \\
- \eta^\alpha (Y) Q X - \eta^\alpha (Y) f P X - \eta^\alpha (Y) f Q X \}
+ f P \nabla_X Y + f Q \nabla_X Y + th (X, Y) + nh (X, Y).
\]

Equations (4.1)-(4.3) follow by comparing the horizontal, vertical and normal components.

Lemma 4.2. Let \( M \) be a \( CR-\)submanifold of an \( S-\)manifold \( \widetilde{M} \) with semi-symmetric non-metric connection. Then,
\[
-A_{fW} Z - f P \nabla_Z W - th (Z, W) = \sum_{\alpha=1}^{s} g (Z, W) \xi_\alpha, \quad (4.4)
\]
\[
\nabla_Z^f f W = f Q \nabla_Z W + nh (Z, W) \quad (4.5)
\]
for any \( Z, W \in \Gamma (D^\perp) \).

Proof. From (3.9), we have
\[
(\nabla_Z f) W = \sum_{\alpha=1}^{s} \{ g (f Z, f W) \xi_\alpha + \eta_\alpha (W) (f^2 Z - f Z) \}
\]
for any \( Z, W \in \Gamma (D^\perp) \). Since \( \eta_\alpha (W) = 0 \) for \( W \in \Gamma (D) \), using (2.2) we get
\[
(\nabla_Z f) W = \sum_{\alpha=1}^{s} g (f Z, f W) \xi_\alpha = \sum_{\alpha=1}^{s} g (Z, W) \xi_\alpha.
\]
Therefore
\[
\nabla_Z f W - f \nabla_Z W = \sum_{\alpha=1}^{s} g (Z, W) \xi_\alpha.
\]
In above equation, using (3.14) and (3.15), we have
\[
-A_{fW} Z + \nabla_Z^f f W - f \nabla_Z W - f h (Z, W) = \sum_{\alpha=1}^{s} g (Z, W) \xi_\alpha
\]
\[
-A_{fW} Z + \nabla_Z^f f W - f P \nabla_Z W - f Q \nabla_Z W - th (Z, W) - nh (Z, W)
= \sum_{\alpha=1}^{s} g (Z, W) \xi_\alpha.
\]
Now comparing tangent and normal parts in above equation, we obtain (4.4) and (4.5). \( \Box \)
Lemma 4.3. Let $M$ be a CR–submanifold of an $S$–manifold $\tilde{M}$ with semi-symmetric non-metric connection. Then,

$$\nabla_X fY - fP\nabla_X Y = \sum_{\alpha=1}^s g(X, Y) \xi_\alpha + th(X, Y),$$  \hspace{1cm} (4.6)

$$h(X, fY) = fQ\nabla_X Y + nh(X, Y)$$  \hspace{1cm} (4.7)

for any $X, Y \in \Gamma(D)$.

Proof. From (3.9), we have

$$(\nabla_X f)Y = \sum_{\alpha=1}^s \{ g(fX, fY) \xi_\alpha + \eta^\alpha(Y)(f^2X - fX) \}$$

for any $X, Y \in \Gamma(D)$. Using $\eta^\alpha(Y) = 0$ for each $Y \in \Gamma(D)$ and (2.2) we obtain

$$(\nabla_X f)Y = \sum_{\alpha=1}^s g(fX, fY) \xi_\alpha = \sum_{\alpha=1}^s g(X, Y) \xi_\alpha.$$

Moreover, we have

$$\nabla_X fY - f\nabla_X Y = \sum_{\alpha=1}^s g(X, Y) \xi_\alpha.$$

Now using (3.14), we have

$$\nabla_X fY + h(X, fY) - f\nabla_X Y - fh(X, Y) = \sum_{\alpha=1}^s g(X, Y) \xi_\alpha$$

$$\nabla_X fY + h(X, fY) - fP\nabla_X Y - fQ\nabla_X Y - th(X, Y) - nh(X, Y)$$

$$= \sum_{\alpha=1}^s g(X, Y) \xi_\alpha.$$

Now comparing tangent and normal parts, we obtain (4.6) and (4.7). \hfill \Box

Lemma 4.4. Let $M$ be a CR–submanifold of an $S$–manifold $\tilde{M}$ with semi-symmetric non-metric connection. Then,

$$\nabla_X \xi_\alpha = -fPX + X, \hspace{1cm} \forall X \in \Gamma(TM)$$  \hspace{1cm} (4.8)

$$h(X, \xi_\alpha) = -fQX, \hspace{1cm} \forall X \in \Gamma(TM)$$  \hspace{1cm} (4.9)

$$A_V \xi_\alpha \in D^\perp, \hspace{1cm} \forall V \in \Gamma(T^\perp M)$$  \hspace{1cm} (4.10)
Proof. Using (3.14) in (3.10), we easily obtain
\[ \nabla_X \xi_\alpha = -fX + X \Rightarrow \nabla_X \xi_\alpha + h(X, \xi_\alpha) = -fX + X \]
which gives
\[ \nabla_X \xi_\alpha + h(X, \xi_\alpha) = -fPX - fQX + X. \]
Now comparing tangent and normal parts, we get
\[ \nabla_X \xi_\alpha = -fPX + X \text{ and } h(X, \xi_\alpha) = -fQX. \]
On the other hand, using (3.3) we have
\[ g(AV\xi_\alpha, X) = g(h(X, \xi_\alpha), V) = g(0, V) = 0 \]
for \( X \in \Gamma(D) \) and \( V \in \Gamma(T^\perp M) \). Using (4.9) in the above equation, we get
\[ g(AV\xi_\alpha, X) = 0, \quad \forall X \in \Gamma(D) \text{ which leads to } AV\xi_\alpha \in \Gamma(D^\perp) \]
also
\[ g(AV\xi_\alpha, X) = 0, \quad \forall X \in \Gamma(D) \Rightarrow g(AV\xi_\alpha, X) = \eta_\alpha(AVX) = 0 \]
which gives (4.10).

Theorem 4.1. Let \( M \) be a CR–submanifold of a S-manifold \( \widetilde{M} \) with semi-symmetric non-metric connection. Then the distribution \( D \) is not integrable.

Proof. For any \( X, Y \in \Gamma(D) \), we have
\[ g([X,Y], \xi_i) = -g(Y, \nabla_X \xi_i) + g(X, \nabla_Y \xi_i). \]
Using (3.10) and (3.14), we have
\[ g([X,Y], \xi_i) = -g(Y, \nabla_X \xi_i - X) + g(X, \nabla_Y \xi_i - Y) \]
\[ = -g(Y, fX) + g(X, fY). \]
Thus \( D \) is integrable if and only if \( g(X, fY) = g(Y, fX) \). From (2.3), the proof is complete. \( \square \)

Theorem 4.2. Let \( M \) be a CR–submanifold of an S–manifold \( \widetilde{M} \) with semi-symmetric non-metric connection. The distribution \( D \oplus Sp\{\xi_1, \ldots, \xi_s\} \) is integrable if and only if
\[ h(X, fY) = h(Y, fX) \]
for any \( X, Y \in \Gamma(D \oplus Sp\{\xi_1, \ldots, \xi_s\}) \).

Proof. From (4.7), we have
\[ h(X, fPY) = fQX + nh(X, Y), \quad \forall X, Y \in \Gamma(D \oplus sp\{\xi_1, \ldots, \xi_s\}). \]
Interchanging \( X \) and \( Y \), we have
\[ h(Y, fPX) = fQY + nh(Y, X), \quad \forall X, Y \in \Gamma(D \oplus sp\{\xi_1, \ldots, \xi_s\}). \]
Adding (4.11) and (4.12), we obtain
\[ h(X, fY) - h(Y, fX) = fQ[X, Y]. \]
Then we have $[X, Y] \in \Gamma(D \oplus sp\{\xi_1, ..., \xi_s\})$ if and only if $h(X, fY) = h(Y, fX)$.

**Corollary 4.1.** Let $M$ be a CR–submanifold of an $S$–manifold $\tilde{M}$ with semi-symmetric non-metric connection. The distribution $D \oplus Sp\{\xi_1, ..., \xi_s\}$ is integrable if and only if

$$A_N fX = -fA_N X$$

for any $X \in \Gamma(D \oplus sp\{\xi_1, ..., \xi_s\})$.

**Definition 4.1.** A CR–submanifold is said to be mixed totally geodesic if $h(X, Z) = 0$, for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$.

**Lemma 4.5.** Let $M$ be a CR–submanifold of an $S$–manifold $\tilde{M}$ with semi-symmetric non-metric connection. Then $M$ is mixed totally geodesic if and only if one of the following satisfied:

$$A_V X \in D \quad (\forall X \in \Gamma(D), \ V \in \Gamma(T^\perp M)),$$

$$A_V X \in D^\perp \quad (\forall X \in \Gamma(D^\perp), \ V \in \Gamma(T^\perp M)).$$

**Proof.** For $X \in \Gamma(D), \ V \in \Gamma(T^\perp M)$ and $Y \in \Gamma(D^\perp)$, consider $A_V X$, then from (3.3) we get

$$g(A_V X, Y) = g(h(X, Y), V)$$

$$= 0 \Leftrightarrow A_V X \in \Gamma(D).$$

Hence, we have

$$g(h(X, Y), V) = 0 \Leftrightarrow h(X, Y) = 0$$

$$\Leftrightarrow A_V X \in \Gamma(D) \quad \forall X \in \Gamma(D), \ V \in \Gamma(T^\perp M),$$

which gives (4.13). In a similar way is deduced relation (4.14).

**Definition 4.2.** The horizontal (resp. vertical) distribution on $D$ (resp. $D^\perp$) is said to be parallel with respect to the connection $\nabla$ on $M$ if

$$\nabla_X Y \in \Gamma(D) \quad (\text{resp. } \nabla_X W \in \Gamma(D^\perp))$$

for any $X, Y \in \Gamma(D)$ (resp. $Z, W \in \Gamma(D^\perp)$).

**Theorem 4.3.** Let $M$ be a $\xi_a$–horizontal CR–submanifold of an $S$–manifold $\tilde{M}$ with semi-symmetric non-metric connection. Then, the horizontal distribution $D$ is parallel if and only if

$$h(X, fY) = h(Y, fX) = fh(X, Y)$$

for all $X, Y \in \Gamma(D)$.

**Proof.** Since every parallel is involutive then the first equality follows immediately. Now since $D$ is parallel, we have

$$\nabla_X fY \in \Gamma(D), \quad \forall X, Y \in \Gamma(D).$$
Then from (4.2), we have

\[\text{th}(X, Y) = 0 \quad \forall X, Y \in \Gamma(D). \tag{4.16}\]

From (4.3), \( D \) is parallel if and only if

\[h(X, fY) = nh(X, Y). \]

But we have

\[fh(X, Y) = \text{th}(X, Y) + nh(X, Y),\]

and from (4.9), \( fh(X, Y) = nh(X, Y) \), which completes the proof. \(\square\)

**Theorem 4.4.** Let \( M \) be a \( CR \)-submanifold of an \( S \)-manifold \( \tilde{M} \) with semi-symmetric non-metric connection. The distribution \( D^\perp \oplus Sp\{\xi_1, ..., \xi_s\} \) is integrable if and only if

\[A_fX - A_fY = \sum_{\alpha=1}^{s} \{\eta^\alpha(X)Y - \eta^\alpha(Y)X\} \tag{4.17}\]

for all \( X, Y \in \Gamma(D^\perp \oplus sp\{\xi_1, ..., \xi_s\})\).

**Proof.** If \( X, Y \in \Gamma(D^\perp \oplus sp\{\xi_1, ..., \xi_s\}) \), then from (4.1) and (4.2) we have

\[-PA_{fQY}X - fP\nabla_X Y = 0, \tag{4.18}\]

\[-QA_{fQY}X - \text{th}(X, Y) = -\sum_{\alpha=1}^{s} \eta^\alpha(Y)X. \tag{4.19}\]

Adding (4.18) and (4.19), we have

\[-A_{fY}X - fP\nabla_X Y - \text{th}(X, Y) = -\sum_{\alpha=1}^{s} \eta^\alpha(Y)X. \tag{4.20}\]

Now interchanging \( X \) and \( Y \), we have

\[-A_{fX}Y - fP\nabla_Y X - \text{th}(X, Y) = -\sum_{\alpha=1}^{s} \eta^\alpha(X)Y. \tag{4.21}\]

Subtracting (4.20) and (4.21), we obtain

\[-A_{fY}X + A_{fX}Y - fP[X, Y] = \sum_{\alpha=1}^{s} \{-\eta^\alpha(Y)X + \eta^\alpha(X)Y\}. \]

Hence \( P[X, Y] = 0 \), we obtain

\[\leftrightarrow A_{fX}Y - A_{fY}X = \sum_{\alpha=1}^{s} \{\eta^\alpha(X)Y - \eta^\alpha(Y)X\}. \]

Therefore \( D^\perp \) is integrable \(\leftrightarrow (4.17) \) holds. \(\square\)
Corollary 4.2. Let $M$ be CR-submanifold of an S-manifold $\widetilde{M}$ with semi-symmetric non-metric connection. Then, the distribution $D^{\perp}$ is integrable if and only if

$$A_{fY}X = A_{fX}Y$$

(4.22)

for all $X, Y \in \Gamma(D^{\perp})$.

Proof. The proof can be obtained directly from (4.17). \hfill \Box

Lemma 4.6. Let $M$ be a CR-submanifold of an S-manifold $\widetilde{M}$ with semi-symmetric non-metric connection. Then, the distribution $D^{\perp}$ is parallel if and only if

$$-A_{fW}Z = \sum_{\alpha=1}^{s} g(Z, W) \xi_{\alpha} + th(Z, W)$$

(4.23)

for all $Z, W \in \Gamma(D^{\perp})$.

Proof. From (4.4), we have

$$-A_{fW}Z - fP\nabla Z W = \sum_{\alpha=1}^{s} g(X, Y) \xi_{\alpha} + th(Z, W) \forall Z, W \in \Gamma(D^{\perp}).$$

If $D^{\perp}$ is parallel then we get

$$\nabla Z W \in \Gamma(D^{\perp}) \Leftrightarrow P\nabla W = 0,$$

which gives (4.23). \hfill \Box

Lemma 4.7. Let $M$ be a CR-submanifold of an S-manifold $\widetilde{M}$ with semi-symmetric non-metric connection. Then the distribution $D^{\perp}$ is parallel if and only if

$$A_{fW}Z \in \Gamma(D^{\perp})$$

(4.24)

for any $Z, W \in \Gamma(D^{\perp})$.

Proof. For any $Z, W \in \Gamma(D^{\perp})$, from (3.9) we have

$$\left(\nabla Z f\right) W = \sum_{\alpha=1}^{s} \left\{g(fZ, fW) \xi_{\alpha} + \eta^{\alpha}(W) (f^{2}Z - fZ)\right\}.$$  

Using (3.14) and (3.15) we obtain

$$\nabla Z fW - f\nabla Z W$$

$$= \sum_{\alpha=1}^{s} \left\{g(fZ, fW) \xi_{\alpha} + \eta^{\alpha}(W) (f^{2}Z - fZ)\right\}$$

$$- A_{fW}Z + \nabla_{Z} fW - f\nabla Z W - fh(Z, W)$$

$$= \sum_{\alpha=1}^{s} \left\{g(fZ, fW) \xi_{\alpha} + \eta^{\alpha}(W) (f^{2}Z - fZ)\right\}.$$
Taking inner product with $Y \in \Gamma(D)$ in the above equation, we have

$$g(-A_{fW}Z, Y) + g \left( \nabla^2_Z fW, Y \right) - g \left( f\nabla_Z W, Y \right) - g \left( f h(Z, W), Y \right) = \sum_{\alpha=1}^{s} \left\{ g \left( fZ, fW \right) g \left( \xi_{\alpha}, Y \right) + \eta^\alpha \left( W \right) g \left( f^2 Z, Y \right) - \eta^\alpha \left( W \right) g \left( fZ, Y \right) \right\}.$$ 

Then we have

$$-g(A_{fW}Z, Y) = g(f\nabla_Z W, Y) = -g(\nabla_Z W, fY).$$

This imply that

$$g \left( A_{fW}Z, Y \right) = 0 \Leftrightarrow A_{fW}Z \in \Gamma(D^\perp).$$

Therefore we obtain

$$\nabla_Z W \in D^\perp \Leftrightarrow A_{fW}Z \in \Gamma(D^\perp) \quad \forall Z, W \in \Gamma(D^\perp).$$

\[ \square \]

References


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