SEMI-PARALLEL TENSOR PRODUCT SURFACES IN SEMI-EUCLIDEAN SPACE $\mathbb{E}^4_2$

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ABSTRACT. In this article, the tensor product surfaces are studied that arise from taking the tensor product of a unit circle centered at the origin in Euclidean plane $\mathbb{E}^2$ and a non-null, unit planar curve in Lorentzian plane $\mathbb{E}^2_1$. Also we have shown that the tensor product surfaces in 4-dimensional semi-Euclidean space with index 2, $\mathbb{E}^4_2$, satisfying the semi-parallelity condition $\mathcal{R}(X,Y)h=0$ if and only if the tensor product surface is a totally geodesic surface in $\mathbb{E}^4_2$.

1. INTRODUCTION

B. Y. Chen initiated the study of the tensor product immersion of two immersions of a given Riemannian manifold [6]. This concept originated from the investigation of the quadratic representation of submanifold. Inspired by Chen’s definition, F. Decruyenaere, F. Dillen, L. Verstraelen and L. Vrancken studied in [8] the tensor product of two immersions of, in general, different manifolds. Under some conditions, this realizes an immersion of the product manifold.

Let $M$ and $N$ be two differentiable manifolds and assume that

$$f : M \to \mathbb{E}^m,$$

and

$$g : N \to \mathbb{E}^n$$

are two immersions. Then the direct sum and tensor product maps are defined respectively by

$$f \oplus h : M \times N \to \mathbb{E}^{m+n}$$

$$(p,q) \to f(p) \oplus h(q) = (f^1(p), \ldots, f^m(p), h^1(q), \ldots, h^n(q))$$

and

$$f \otimes h : M \times N \to \mathbb{E}^{mn}$$

$$(p,q) \to f(p) \otimes h(q) = (f^1(p)h^1(q), \ldots, f^1(p)h^n(q), \ldots, f^m(p)h^n(q))$$

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Necessary and sufficient conditions for $f \otimes h$ to be an immersion were obtained in [7]. It is also proved there that the pairing $(\otimes, \otimes)$ determines a structure of a semiring on the set of classes of differentiable manifolds transversally immersed in Euclidean spaces, modulo orthogonal transformations. Some semirings were studied in [8]. In the special case, a tensor product surface is obtained by taking the tensor product of two curves. In many papers, minimality and totally reality properties of a tensor product surfaces were studied for example [2], [10], [11], [12]. The relations between a tensor product surface and a Lie group was shown in [15], [16]. In [2], Bulca and Arslan studied tensor product surfaces in 4-dimensional Euclidean space $\mathbb{E}^4$ and they show that tensor product surfaces satisfying the semi-parallelity condition $\mathcal{R}(X,Y), h = 0$ are totally umbilical surface.

In this article, we investigate a tensor product surface $M$ which is obtained from two curves. One of them is a unit circle centered at the origin in Euclidean plane $\mathbb{E}^2$ and a non-null, unit planar curve in Lorentzian plane $\mathbb{E}^2_{1}$. Firstly, we investigated some geometric properties of the tensor product surface in pseudo-Euclidean 4-space $\mathbb{E}^4_2$ then we obtain the sufficient and necessary conditions for the surface satisfying the semi parallelity condition $\mathcal{R}(X,Y), h = 0$.

We remark that the notions related with pseudo-Riemannian geometry are taken from [14].

2. Preliminaries

In the present section we give some definitons about Riemannian submanifolds from [5] and [4]. Let $\iota : M \rightarrow \mathbb{E}^n$ be an immersion from an $m-$dimensional connected Riemannian manifold $M$ into an $n-$dimensional Euclidean space $\mathbb{E}^n$. We denote by $g$ the metric tensor of $\mathbb{E}^n$ as well as induced metric on $M$. Let $\nabla$ be the Levi-Civita connection of $\mathbb{E}^n$ and $\tilde{\nabla}$ the induced connection on $M$. Then the Gaussian and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

(2.1)

$$\tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N$$

where $X, Y$ are vector fields tangent to $M$ and $N$ is normal to $M$. Moreover, $h$ is the second fundamental form, $\nabla^\perp$ is linear connection induced in the normal bundle $T^\perp M$, called normal connection and $A_N$ is the shape operator in the direction of $N$ that is related with $h$ by

$$< h(X,Y), N >= < A_N X, Y > .$$

(2.2)

If the set $\{X_1, ..., X_m\}$ is a local basis for $\chi(M)$ and $\{N_1, ..., N_{n-m}\}$ is an orthonormal local basis for $\chi^\perp(M)$, then $h$ can be written as

$$h = \sum_{\alpha=1}^{n-m} \sum_{i,j=1}^{m} h_{ij}^\alpha N_\alpha ,$$

(2.3)
where
\[ h_{ij}^\alpha = \langle h(X_i, X_j), N_\alpha \rangle. \]

The covariant differentiation \( \nabla h \) of the second fundamental form \( h \) on the direct sum of the tangent bundle and the normal bundle \( TM \oplus T^\perp M \) of \( M \) is defined by
\[
(\nabla_X h)(Y, Z) = \nabla_X^h h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),
\]
for any vector fields \( X, Y \) and \( Z \) tangent to \( M \). Then we have the Codazzi equation as
\[
(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z). \tag{2.5}
\]

We denote by \( R \) the curvature tensor associated with \( \nabla \);
\[
R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z, \tag{2.6}
\]
and denote by \( R^\perp \) the curvature tensor associated with \( \nabla^\perp \)
\[
R^\perp(X, Y)\eta = \nabla^\perp_X \nabla^\perp_Y \eta - \nabla^\perp_Y \nabla^\perp_X \eta - \nabla^\perp_{[X,Y]} \eta. \tag{2.7}
\]

The equations Gauss and Ricci are given by
\[
\langle R(X, Y)Z, W \rangle = \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle, \tag{2.8}
\]
\[
\langle \hat{R}(X, Y)\eta, \xi \rangle - \langle R^\perp(X, Y)\eta, \xi \rangle = \langle [A_\eta, A_\xi]X, Y \rangle, \tag{2.9}
\]
for any vector fields \( X, Y, Z, W \) tangent to \( M \) and \( \xi, \eta \) normal vector fields to \( M \).

The Gaussian curvature of \( M \) is defined by
\[
K = \langle h(X_1, X_2), h(X_2, X_1) \rangle - \| h(X_1, X_2) \|^2 \tag{2.10}
\]
where the set \( \{X_1, X_2\} \) is a linearly independent subset of \( \chi(M) \).

The normal curvature \( K_N \) of \( M \) is defined by
\[
K_N = \left\{ \sum_{1=\alpha < \beta}^{n-m} \langle R^\perp(X_1, X_2)N_\alpha, N_\beta \rangle^2 \right\}^{1/2} \tag{2.11}
\]
where \( \{N_\alpha, N_\beta\} \) is an orthonormal basis of \( \chi^\perp(M) \). From (2.11) we conclude that
\[ K_N = 0 \text{ if and only if } \nabla^\perp \text{ is a flat normal connection of } M. \]

Further, the mean curvature vector \( \overline{H} \) of \( M \) is defined by
\[
\overline{H} = \frac{1}{m} \sum_{\alpha=1}^{n-m} tr(A_{N_\alpha})N_\alpha \tag{2.12}
\]

Let us consider the product tensor \( \hat{R}, h \) of the curvature tensor \( \overline{R} \) with the second fundamental form \( h \) is defined by
\[
(\hat{R}(X, Y).h)(Z, T) = \nabla_X (\overline{\nabla}_Y h(Z, T)) - \overline{\nabla}_Y (\nabla_X h(Z, T)) - \overline{\nabla}_{[X,Y]} h(Z, T) \tag{2.13}
\]
for all $X, Y, Z, T$ tangent to $M$.

The surface $M$ is said to be semi - parallel (or semi-symmetric) if $	ilde{R}.h = 0$, i.e. $\tilde{R}(X, Y).h = 0$ [9], [17]. It is easily seen that

$$(\tilde{R}(X, Y).h)(Z, T) = R^h(X, Y)h(Z, T) - h(R(X, Y)Z, T) - h(Z, R(X, Y)T)$$  (2.14)

**Lemma 2.1.** [9] Let $M \subset \mathbb{E}^n$ be a smooth surface given with the patch $X(u, v)$. Then the following equalities are hold;

$$(\tilde{R}(X_1, X_2).h)(X_1, X_1) = \left( \sum_{\alpha=1}^{n-2} h^\alpha_{11}(h^\alpha_{22} - h^\alpha_{11} + 2K) \right) h(X_1, X_2)$$

$$+ \sum_{\alpha=1}^{n-2} h^\alpha_{12}h^\alpha_{21}(h(X_1, X_1) - h(X_2, X_2))$$

$$(\tilde{R}(X_1, X_2).h)(X_1, X_2) = \left( \sum_{\alpha=1}^{n-2} h^\alpha_{12}(h^\alpha_{22} - h^\alpha_{11}) \right) h(X_1, X_2)$$

$$+ \sum_{\alpha=1}^{n-2} h^\alpha_{12}h^\alpha_{21} - K \right) (h(X_1, X_1) - h(X_2, X_2))$$

$$(\tilde{R}(X_1, X_2).h)(X_2, X_2) = \left( \sum_{\alpha=1}^{n-2} h^\alpha_{22}(h^\alpha_{22} - h^\alpha_{11} - 2K) \right) h(X_1, X_2)$$

$$+ \sum_{\alpha=1}^{n-2} h^\alpha_{22}h^\alpha_{21}(h(X_1, X_1) - h(X_2, X_2))$$  (2.15)

**Semi parallel surfaces classified by J. Deprez [9].**

**Theorem 2.1.** [9] Let $M$ be a surface in $n$-dimensional Euclidean space $\mathbb{E}^n$. Then $M$ is semi-parallel if and only if locally;

i) $M$ is equivalent to 2-sphere, or

ii) $M$ has trivial normal connection, or

iii) $M$ is an isotropic surface in $\mathbb{E}^5 \subset \mathbb{E}^n$ satisfying $\|H\|^2 = 3K$.

3. Tensor product surfaces of a Euclidean plane curve and a Lorentzian plane curve

Minimal and pseudo-minimal tensor product surfaces of a Lorentzian plane curve and a Euclidean plane curve was studied by I. Mihai and et al. in [13]. They also gave some examples of non-minimal pseudo-umbilical tensor product surfaces. It is well known that the tensor product of two immersions is not commutative. Thus the tensor product surfaces of a Euclidean plane curve and a Lorentzian plane curve is a new surface in 4-dimensional semi-Euclidean space with index 2.

In the following section, we will consider the tensor product immersions which is obtained from a Euclidean plane curve and a Lorentzian plane curve. Let $c_1 : \mathbb{R} \to \mathbb{E}^2$ be a Euclidean plane curve and $c_2 : \mathbb{R} \to \mathbb{E}^4_2$ be a non-null Lorentzian plane curve. Put $c_1(t) = (\alpha_1(t), \alpha_2(t))$ and $c_2(s) = (\beta_1(s), \beta_2(s))$.

Then their tensor product surface is given by

$$x = c_1 \otimes c_2 : \mathbb{R}^2 \to \mathbb{E}^4_2$$
\[
    x(t, s) = (\alpha_1(t)\beta_1(s), \alpha_1(t)\beta_2(s), \alpha_2(t)\beta_1(s), \alpha_2(t)\beta_2(s)).
\]

The metric tensor on \(E^2\) and \(E^4\) is given by
\[
    g = -dx_1^2 + dx_2^2
\]
and
\[
    g = -dx_1^2 + dx_2^2 - dx_3^2 + dx_4^2,
\]
respectively.

If we take \(c_1\) as a Euclidean unit circle \(c_1(t) = (\cos t, \sin t)\) at centered origin
and \(c_2(s) = (\alpha(s), \beta(s))\) is a spacelike or timelike curve with unit speed then the
surface patch becomes
\[
    M : x(t, s) = (\alpha(s)\cos t, \beta(s)\cos t, \alpha(s)\sin t, \beta(s)\sin t) \tag{3.1}
\]
An orthonormal frame tangent to \(M\) is given by
\[
    e_1 = \frac{1}{\|c_2\|} \frac{\partial x}{\partial t} = \frac{1}{\|c_2\|} (-\alpha(s)\sin t, -\beta(s)\sin t, \alpha(s)\cos t, \beta(s)\cos t), \tag{3.2}
\]
\[
    e_2 = \frac{\partial x}{\partial s} = (\alpha'(s)\cos t, \beta'(s)\cos t, \alpha'(s)\sin t, \beta'(s)\sin t).
\]
The normal space of \(M\) is spanned by
\[
    n_1 = (\beta'(s)\cos t, \alpha'(s)\cos t, \beta'(s)\sin t, \alpha'(s)\sin t), \tag{3.3}
\]
\[
    n_2 = \frac{1}{\|c_2\|} (-\beta(s)\sin t, -\alpha(s)\sin t, \beta(s)\cos t, \alpha(s)\cos t)
\]
where
\[
    g(e_1, e_1) = -g(n_2, n_2) = \frac{g(c_2(s), c_2(s))}{\|c_2\|^2} = \varepsilon_1, \tag{3.4}
\]
\[
    g(e_2, e_2) = -g(n_1, n_1) = g(c_2(s), c_2'(s)) = \varepsilon_2
\]
and \(\varepsilon_1 = \mp 1, \varepsilon_2 = \mp 1\).
By covariant differentiation with respect to $e_1$ and $e_2$ a straightforward calculation gives

$$
\nabla_{e_1} e_1 = a e_2 e_2 - b e_2 n_1
$$
$$
\nabla_{e_1} e_2 = -a e_1 e_1 - b e_1 n_2
$$
$$
\nabla_{e_1} n_1 = -b e_1 e_1 - a e_1 n_2
$$
$$
\nabla_{e_1} n_2 = -b e_2 e_2 + a e_2 n_1
$$

where $a$, $b$ and $c$ are Christoffel symbols and as follows

$$
a = a(s) = \frac{aa' - \beta \beta'}{||e_2||^2},
$$
$$
b = b(s) = \frac{\alpha \beta' - \alpha' \beta}{||e_2||^2},
$$
$$
c = c(s) = \alpha'' \beta - \alpha' \beta'.
$$

In addition, from (2.3) second fundamental form of this structure is written as,

$$
h = \sum_{i,j,a=1}^{2} \varepsilon_{ij} h_{ij}^a n_a,
$$

where

$$
h_{11} = b \\
h_{12} = h_{21} = 0 \\
h_{22} = c \\
h_{21} = h_{12} = h_{22} = 0
$$

By considering equations (3.8) and 3.9, we conclude that

**Corollary 3.1.** If $b = 0$ then $c$ is also zero.

Also by using Corollary 3.1 and (3.11), we have

**Corollary 3.2.** $M$ is a totally geodesic surface in $E_4^2$ if and only if $b = 0$ which means that $c_2$ is a straightline passing through the origin.

If $b = 0$, from (3.8), we get $c_2(s) = \beta(s)(\lambda, 1)$. Since $M$ is a non-degenerate surface, the position vector of $c_2$ cannot be a null then $\lambda \neq \pm 1$. In this case, we can write the parametric equation of tensor product surface $M$ as follows

$$
M : x(t, s) = (\lambda \beta(s) \cos t, \beta(s) \cos t, \lambda \beta(s) \sin t, \beta(s) \sin t), \ \lambda \neq \pm 1, \ \lambda \in \mathbb{R}.
$$

Indeed, this surface fully lies in a cone surface passing through the origin (but not light cone) in 4-dimensional semi-Euclidean space with index 2, $E_4^2$, with equation

$$
-x_1^2 + \lambda^2 x_2^2 - x_3^2 + \lambda^2 x_4^2 = 0
$$

where $\lambda \neq \pm 1$ and $\lambda \in \mathbb{R}$. 
The induced covariant differentiation on $M$ as follows,
\[
\begin{align*}
\nabla_{e_1} e_1 &= a \varepsilon_2 e_2, \\
\nabla_{e_1} e_2 &= -a \varepsilon_1 e_1, \\
\nabla_{e_2} e_1 &= 0, \\
\nabla_{e_2} e_2 &= 0,
\end{align*}
\]
(3.12)
\[
\begin{align*}
\nabla_{e_1} n_1 &= a \varepsilon n_2, \\
\nabla_{e_1} n_2 &= a \varepsilon n_1, \\
\nabla_{e_2} n_1 &= 0, \\
\nabla_{e_2} n_2 &= 0,
\end{align*}
\]
(3.13)
where the equalities (3.13) and (3.14) define the normal connection on $M$.

**Lemma 3.1.** Let $x = c_1 \otimes c_2$ be a tensor product immersion of a Euclidean unit circle $c_1$ at centered origin and unit speed non-null Lorentzian curve $c_2$ in $\mathbb{E}_1^2$. Then the shape operators of $M$ in direction of $n_1$ and $n_2$ are given by respectively,
\[
A_{n_1} = \begin{bmatrix} b \varepsilon_1 & 0 \\
0 & c \varepsilon_2 \end{bmatrix}, \quad A_{n_2} = \begin{bmatrix} 0 & b \varepsilon_1 \\
b \varepsilon_2 & 0 \end{bmatrix}.
\]
(3.15)

By a simple calculation, we see that Gauss and Ricci equations of $M$ are identical and they are given by as follow
\[
a' - a^2 \varepsilon_1 = b^2 \varepsilon_1 - b \varepsilon_2,
\]
and Codazzi equation of $M$ is
\[
b' = 2ab \varepsilon_1 - ac \varepsilon_2.
\]
(3.17)
Thus we give the following theorem.

**Theorem 3.1.** If $M$ is a tensor product surface of a Euclidean unit circle at centered origin and a non-null unit speed Lorentzian curve in $\mathbb{E}_1^2$ then the Christoffel symbols of $M$ satisfy the following Riccati equation
\[
(a + b)' = \varepsilon_1 (a + b)^2 - c \varepsilon_2 (a + b).
\]
(3.18)

**Theorem 3.2.** Let $M$ be a tensor product surface given with the surface patch (3.1). Then there exist following relation between Gaussian curvature $K$ and normal curvature $K_N$
\[
K_N = |K| = |b^2 \varepsilon_1 - b \varepsilon_2|.
\]

**Theorem 3.3.** Let $M$ be a tensor product surface given with the surface patch (3.1). Then the followings are equivalent,
\[ \nabla_\perp \text{ is a flat connection,} \]

\[ K_N = K = 0, \]

\[ b = 0 \text{ or } \varepsilon_1 b = \varepsilon_2 c. \]

Now, we suppose that \( M \) is a semi parallel surface, i.e., \( \bar{R}.h = 0 \). From (2.15) we get

\[
\begin{align*}
\varepsilon_1 (c - b + 2\varepsilon_1 - 2c\varepsilon_2) &= 0, \\
\varepsilon_2 (b - \varepsilon_1 + c\varepsilon_2)(c - b) &= 0, \\
\varepsilon_1 (2b^2\varepsilon_1 + bc - c^2 - 2bc\varepsilon_2) &= 0.
\end{align*}
\]

**Theorem 3.4.** Let \( M \) be a tensor product surface given with the surface patch (3.1). Then \( M \) is a semi parallel surface if and only if

i) For \( \varepsilon_1 = \varepsilon_2 \), either \( b = 0 \) or \( b = c \),

ii) For \( \varepsilon_1 \neq \varepsilon_2 \), \( b = 0 \).

**Corollary 3.3.** Let \( M \) be a tensor product surface given with the surface patch (3.1) with \( \varepsilon_1 \neq \varepsilon_2 \) then \( M \) is a semi parallel surface if and only if \( M \) is a a totally geodesic surface in \( E^4 \).

**References**


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