ON COFINITELY WEAK RAD-SUPPLEMENTED MODULES

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Abstract. In this paper, necessary and sufficient conditions for a quotient module are found to be a cofinitely weak Rad-supplemented module under which circumstances. Nevertheless, some relations are investigated between cofinitely Rad-supplemented modules and cofinitely weak Rad-supplemented modules. Lastly, we show that an arbitrary ring $R$ is a left Noetherian $V$–ring if and only if every weak Rad-supplemented $R$–module is injective.

1. Introduction

Throughout the paper, $R$ will be an associative ring with identity, $M$ will be an $R$–module and all modules will be unital left $R$–modules unless otherwise specified. By $N \leq M$, we mean that $N$ is a submodule of $M$. Recall that a submodule $L$ of $M$ is small in $M$ and denoted by $L \ll M$, if $M \neq L + K$ for every proper submodule $K$ of $M$. A submodule $S$ of $M$ is said to be essential in $M$ and denoted by $S \trianglelefteq M$, if $S \cap N \neq 0$ for every nonzero submodule $N \leq M$. We write $\text{Rad}(M)$ for the Jacobson radical of a module $M$. An $R$–module $M$ is called supplemented, if every submodule $N$ of $M$ has a supplement in $M$, i.e. a submodule $K$ is minimal with respect to $M = N + K$. $K$ is supplement of $N$ in $M$ if and only if $M = N + K$ and $N \cap K \ll K$ [16].

If $M = N + K$ and $N \cap K \ll M$, then $K$ and $N$ are called weak supplements of each other. Also $M$ is called a weakly supplemented module if every submodule of $M$ has a weak supplement in $M$ [13, 18]. By using this definition, Büyükaşk and Lomp showed in [6] that a ring $R$ is left perfect if and only if every left $R$–module is weakly supplemented if and only if $R$ is semilocal and the radical of the countably infinite free left $R$–module has a weak supplement. Furthermore Alizade and Büyükaşk showed that a ring $R$ is semilocal if and only if every direct product of simple modules is weakly supplemented [3].

In [17], Xue introduced Rad-supplemented modules. Let $M$ be an $R$– module, $N$ and $K$ be any submodules of $M$ with $M = N + K$. If $N \cap K \leq \text{Rad}(K)$...
(N \cap K \leq \text{Rad}(M))$, then $K$ is called a (weak) Rad-supplement of $N$ in $M$. Besides $M$ is called (weakly) Rad-supplemented module provided that each submodule has a (weak) Rad-supplement in $M$. For characterizations of Rad-supplemented and weak Rad-supplemented modules, we refer to [15] and [17]. Since the Jacobson radical of a module is the sum of all small submodules, every supplement is a Rad-supplement.

Certain modules whose maximal submodules have supplements are studied in [1]. Also in the same paper, cofinitely supplemented modules are introduced. A submodule $N$ of $M$ is said to be cofinite if $\frac{M}{N}$ is finitely generated. $M$ is called cofinitely (weak) supplemented if every cofinite submodule has a (weak) supplement in $M$ [1, 2]. Nevertheless, it is known by [1, Theorem 2.8] and [2, Theorem 2.11] that an $R$–module $M$ is cofinitely (weak) supplemented if and only if every maximal submodule of $M$ has a (weak) supplement in $M$. Clearly, supplemented modules are cofinitely supplemented and weakly supplemented modules are cofinitely weak supplemented ones.

$M$ is called cofinitely Rad-supplemented if every cofinite submodule of $M$ has a Rad-supplement [5]. Since every submodule of a finitely generated module is cofinite, a finitely generated module is Rad-supplemented if and only if it is cofinitely Rad-supplemented. According to [12], if every cofinite submodule of $M$ has a Rad-supplement that is a direct summand of $M$, then $M$ is called a $\oplus$–cofinitely Rad-supplemented module.

In a present paper [10], a module is called cofinitely weak Rad-supplemented if every cofinite submodule has a weak Rad-supplement and totally cofinitely weak Rad-supplemented if every submodule is cofinitely weak Rad-supplemented. Also it is proved in [10] that any arbitrary sum of cofinitely weak Rad-supplemented modules is a cofinitely weak Rad-supplemented module. Clearly this implies that any finite direct sum of cofinitely weak Rad-supplemented modules is also cofinitely weak Rad-supplemented. We will show that an infinite direct sum of totally cofinitely weak Rad-supplemented modules is totally cofinitely weak Rad-supplemented under certain conditions. Also we will prove that every torsion module over a Dedekind domain is a cofinitely weak Rad-supplemented module and find some conditions to show when any module over a Dedekind domain is cofinitely weak Rad-supplemented.

2. Main Results

Following [5], a module $M$ is called $w$–local if it has a unique maximal submodule.

**Theorem 1.** Every $w$–local module is cofinitely weak Rad-supplemented.

**Proof.** Let $M$ be a module and $U$ be a cofinite submodule of $M$. Since $\frac{M}{U}$ is finitely generated, it has a maximal submodule such as $\frac{P}{\tilde{P}}$. Therefore $P$ is a maximal
submodule of $M$. Then we have $U + M = M$ and $U \cap M = U \subseteq P = \text{Rad}(M)$. Hence $M$ is cofinitely weak Rad-supplemented.

Recall that a module $M$ is called refinable (or suitable), if for any submodules $U, V \leq M$ with $U + V = M$, there exists a direct summand $U_1$ of $M$ with $U_1 \leq U$ and $U_1 + V = M$.

**Theorem 2.** Let $M$ be a refinable $R$–module. Then the following are equivalent:
(i) $M$ is $\oplus$–cofinitely Rad-supplemented,
(ii) $M$ is cofinitely Rad-supplemented,
(iii) $M$ is cofinitely weak Rad-supplemented.

*Proof.* The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious.

$(iii) \Rightarrow (i)$ Let $M$ be a cofinitely weak Rad-supplemented module and $N$ be a cofinite submodule of $M$. Then, we have $M = N + K$ and $N \cap K \leq \text{Rad}(M)$ where $K$ is a submodule of $M$. Since $M$ is a refinable module, it has a direct summand $L$ such that $L \leq K$ and $M = L + N$. Following this, $N \cap L \leq N \cap K \leq \text{Rad}(M)$ implies that $L$ is weak Rad-supplement of $N$. By using [14, Proposition 4], we get that $L$ is Rad-supplement of $N$. Therefore, $M$ is $\oplus$–cofinitely Rad-supplemented.

A ring $R$ is called a left $V$–ring if every simple left $R$–module is injective.

**Theorem 3.** For an arbitrary ring $R$, the following are equivalent:
(i) Every weakly Rad-supplemented $R$–module is injective,
(ii) $R$ is a left Noetherian $V$–ring.

*Proof.* $(i) \Rightarrow (ii)$ Assume that $M$ is a $\oplus$–supplemented $R$–module. Since $M$ is weak Rad-supplemented, it is an injective module. By Proposition 5.3 in [11] we get that $R$ is a left Noetherian $V$–ring.

$(ii) \Rightarrow (i)$ Let $M$ be a weakly Rad-supplemented module. Since $R$ is a left Noetherian $V$–ring, we get $\text{Rad}(M) = 0$ by Villamayor theorem in [7]. Then, $M$ is semisimple and so $\oplus$–supplemented. Again using Proposition 5.3 in [11], we obtain $M$ is an injective module.

**Corollary 1.** Let $R$ be a commutative ring. Then, every weakly Rad-supplemented $R$–module is injective if and only if $R$ is semisimple.

*Proof.* Suppose that every weakly Rad-supplemented module is injective. By using the preceding theorem, we can say that $R$ is a left Noetherian $V$–ring. Thus, $R$ is semisimple by Proposition 1 and first corollary of [7]. The other side of the proof is obvious by [16, 20.3].

**Theorem 4.** Let $M$ be a module and $N$ be a submodule of $M$. If every cofinite submodule containing $N$ of $M$ has a weak Rad-supplement in $M$, then $\frac{M}{N}$ is cofinitely weak Rad-supplemented.
Proof. Let $\frac{U}{N}$ be a cofinite submodule of $\frac{M}{N}$. Since $\frac{(U+N)}{N} \cong \frac{M}{U}$, we get that $U$ is a cofinite submodule of $M$ containing $N$. Hence, we can find a submodule $V$ of $M$ such that $M = U + V$ and $U \cap V \leq \text{Rad}(M)$. By using Proposition 3.2 of [15], we can deduce that $\frac{(V+N)}{N}$ is a weak Rad-supplement of $\frac{U}{N}$ in $\frac{M}{N}$. Therefore, $\frac{M}{N}$ is a cofinitely weak Rad-supplemented module.

Remark. While a quotient module of a module is a cofinitely weak Rad-supplemented module, it may not be a cofinitely weak Rad-supplemented module. For example, $\mathbb{Z}/\mathbb{Z}$ isn’t cofinitely weak Rad-supplemented but $\mathbb{Z}_p$ is cofinitely weak Rad-supplemented for any prime number $p$.

Proposition 1. Let $M$ be a cofinitely weak Rad-supplemented $R$–module. Then every Rad-supplement in $M$ is cofinitely weak Rad-supplemented.

Proof. Let $V$ be a Rad–supplement of $U$ in $M$. That means $M = U + V$ and $U \cap V \leq \text{Rad}(V)$. Since $\frac{M}{V} = \frac{U + V}{V} \cong \frac{V}{\text{Rad}(V)}$, we get that $\frac{V}{\text{Rad}(V)}$ is a cofinitely weak Rad-supplemented module by [10, Proposition 6]. Theorem 4 in the same paper implies that $V$ is cofinitely weak Rad-supplemented.

Theorem 5. Let $R$ be a Dedekind domain and $M$ be a torsion $R$–module. Then $M$ is cofinitely weak Rad-supplemented.

Proof. By [3, Corollary 2.7], we have $\frac{M}{\text{Rad}(M)}$ is semisimple and so cofinitely weak Rad-supplemented.

Theorem 6. Let $R$ be a Dedekind domain, $\frac{M}{\text{Rad}(M)}$ be finitely generated and $\text{Rad}(M) \leq M$. If $\text{Rad}(M)$ is cofinitely weak Rad-supplemented, then $M$ is cofinitely weak Rad-supplemented.

Proof. Suppose that $\frac{M}{\text{Rad}(M)}$ is generated by $m_1 + \text{Rad}(M), m_2 + \text{Rad}(M), \ldots, m_n + \text{Rad}(M)$. Then, for finitely generated submodule $K = Rm_1 + Rm_2 + \ldots + Rm_n$, we have $M = \text{Rad}(M) + K$ and $K \cap \text{Rad}(M)$ is finitely generated as $K$ is finitely generated. So $K \cap \text{Rad}(M) \ll M$ by Lemma 2.3 in [3]. That is to say, $K$ is a weak supplement of $\text{Rad}(M)$ of $M$. Since $\text{Rad}(M) \leq M$, we get $\frac{M}{\text{Rad}(M)}$ is torsion. Besides this, Proposition 9.15 of [4] implies that $\text{Rad}\left(\frac{M}{\text{Rad}(M)}\right) = 0$. Hence $\frac{M}{\text{Rad}(M)}$ is semisimple by Corollary 2.7 in [3]. If we consider $0 \to \text{Rad}(M) \to M \to \frac{M}{\text{Rad}(M)} \to 0$, then $M$ is cofinitely weak Rad-supplemented by Theorem 7 in [10].

Proposition 2. Let $R$ be a non-semilocal commutative domain. If $M$ is totally cofinitely weak Rad-supplemented, then $M$ is torsion.

Proof. Suppose that $\text{Ann}(m) = 0_R$ for some $m \in M$. Then we have $Rm \cong R$. Since $Rm$ is cofinitely weak Rad-supplemented, $R$ is also (cofinitely) weak Rad-supplemented. Then by 17.2 of [8], $R$ is a semilocal ring which gives a contradiction. Thus, $M$ is a torsion module.
Theorem 7. Let $R$ be an arbitrary ring and $M = \bigoplus_{i \in I} M_i$ such that $M_i$ is totally cofinitely weak Rad-supplemented for all $i \in I$. If $U = \bigoplus_{i \in I} (U \cap M_i)$ for every submodule $U$ of $M$, then $M$ is totally cofinitely weak Rad-supplemented.

Proof. Assume that $U$ is a submodule of $M$ and $V$ is a cofinite submodule of $U$ where $U = \bigoplus_{i \in I} (U \cap M_i)$. Since $V = \bigoplus_{i \in I} (V \cap M_i)$ and $\frac{U}{V} \cong \bigoplus_{i \in I} \left( \frac{U}{V} \cap M_i \right)$, we get that $V \cap M_i$ is a cofinite submodule of $U \cap M_i$ for all $i \in I$. We know that $U \cap M_i$ is cofinitely weak Rad-supplemented. Therefore $V \cap M_i$ has a weak Rad-supplement $K_i$ in $U \cap M_i$ for all $i \in I$. Let $K = \bigoplus_{i \in I} K_i$. Then we obtain $U = V + K$ and $V \cap K \leq \text{Rad}(U)$. As a result, $U$ is cofinitely weak Rad-supplemented and so $M$ is totally cofinitely weak Rad-supplemented.

Let $R$ be a Dedekind domain and $M$ be an $R$–module. By $\Omega$, we denote the set of all maximal ideals of $R$. The submodule $T_P(M) = \{ m \in M \mid P^n m = 0 \text{ for some } n \geq 1 \}$ is called the $P$–primary part of $M$.

Theorem 8. Let $R$ be a non-semilocal Dedekind domain. Then, $M$ is a totally cofinitely weak Rad–supplemented module if and only if $M$ is torsion and $T_P(M)$ is totally cofinitely weak Rad-supplemented for every $P \in \Omega$.

Proof. Assume that $M$ is a totally cofinitely weak Rad-supplemented module. Then $M$ is torsion by Proposition 2. On the other hand $T_P(M)$ is totally cofinitely weak Rad-supplemented for every $P \in \Omega$. Because every submodule of a totally cofinitely weak Rad-supplemented module is a totally cofinitely weak Rad-supplemented module.

Conversely, we can write $M = \bigoplus_{P \in \Omega} T_P(M)$ by Proposition 6.9 in [9]. Let $N$ be a submodule of $M$. Since $M$ is torsion, $N$ is also a torsion module. By using the same proposition, we can write that $N = \bigoplus_{P \in \Omega} T_P(N)$. Therefore, $\bigoplus_{P \in \Omega} (N \cap T_P(M))$ and $T_P(M)$ is totally cofinitely weak Rad-supplemented for every $P \in \Omega$. As a result, $M$ is totally cofinitely weak Rad-supplemented by the preceding theorem.

Theorem 9. Any torsion module over a Dedekind domain is totally cofinitely weak Rad–supplemented.

Proof. Let $R$ be a Dedekind domain, $M$ be a torsion $R$–module and $N$ be a submodule of $M$. Due to Corollary 2.7 of [3], $\frac{N}{\text{Rad}(N)}$ is semisimple and so it is cofinitely weak Rad–supplemented. Therefore $N$ is cofinitely weak Rad–supplemented by Theorem 4 of [10].

References


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