ON MULTIPLICATIVE (GENERALIZED)-DERIVATIONS IN SEMIPRIME RINGS

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Abstract. In this paper, we study commutativity of a prime or semiprime ring using a map $F: R \to R$, multiplicative (generalized)-derivation and a map $H: R \to R$, multiplicative left centralizer, under the following conditions: For all $x, y \in R$, i) $F(xy) + H(xy) = 0$, ii) $F(xy) + H(yx) = 0$, iii) $F(x)F(y) + H(xy) = 0$, iv) $F(xy) + H(yx) \in Z$, v) $F(xy) + H(yx) \in Z$, vi) $F(x)F(y) + H(xy) \in Z$.

1. Introduction

Let $R$ be a ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ (resp. $x \circ y$) means that $xy - yx$ (resp. $xy + yx$). We use many times the commutator identities $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$, for all $x, y, z \in R$. Recall that $R$ is prime if for any $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$ and $R$ is semiprime if for any $a \in R$, $aRa = (0)$ implies $a = 0$. Therefore, it is known that if $R$ is semiprime, then $aRb = (0)$ yields $ab = 0$ and $ba = 0$. In [3], Bresar was introduced the generalized derivation as the following: Let $F: R \to R$ be an additive map and $g: R \to R$ be a derivation. If $F(xy) = F(x)y + xg(y)$ holds for all $x, y \in R$, then $F$ is called a generalized derivation associated with $g$. It is symbolized by $(F, g)$. Hence the concept of generalized derivation involves the concept of derivation. In [4] Daif defined multiplicative derivation as the following. Let $D: R \to R$ be a map. If $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$, then $D$ is said to be multiplicative derivation. Thus the concept of multiplicative derivation involves the concept of derivation. Next, in [5], Daif and El-Sayiad gave multiplicative generalized derivation as the following. Let $F: R \to R$ be a map and $d: R \to R$ be a derivation. If $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, then $F$ is called a multiplicative generalized derivation associated with $d$. Hence the concept of multiplicative generalized derivation involves the concept of generalized derivation. Let $H: R \to R$ be a map. If $H(xy) = H(x)y$ holds for all
holds for all \( x, y \in R \), then \( H \) is called a multiplicative left centralizer ([6]). In [11], Dhara and Ali gave definition of multiplicative (generalized)-derivation as the following. Let \( F, f : R \to R \) be two maps. If for all \( x, y \in R \), \( F(xy) = F(x)y + xf(y) \), then \( F \) is called a multiplicative (generalized)-derivation associated with \( f \). Hence the concept of multiplicative (generalized)-derivation involves the concept of multiplicative generalized derivation.

With the generalization of derivation, it is given following conditions of commutativity of prime or semiprime ring. As a first time, in Ashraf and Rehman’s paper [7], if \( d(xy) \pm xy \in Z(R) \) holds for all \( x, y \in I \), then \( R \) is commutative where \( R \) is a prime ring, \( I \) is nonzero two sided ideal of \( R \) and \( d : R \to R \) is a derivation. In papers ([8], [12], [9], [11], [1], [10], [14]), studied following conditions. \( i) \) \( F(xy) \pm xy \in Z(R), F(xy) \pm yx \in Z(R), F(x)F(y) \pm xy \in Z(R) \) for all \( x, y \in I \), where \( R \) is a prime ring, \( I \) is a nonzero two sided ideal of \( R, d : R \to R \) is a derivation and \( F : R \to R \) is a generalized derivation ([8]). \( ii) \) \( d([x, y]) = \pm [x, y] \) for all \( x, y \in I \), where \( R \) is a semiprime ring, \( I \) is a nonzero ideal of \( R \) and \( d : R \to R \) is a derivation. ([9]). \( iii) \) \( F([x, y]) = \pm [x, y] \) for all \( x, y \in I \), where \( R \) is a prime ring, \( I \) is a nonzero two sided ideal of \( R \) and \( d : R \to R \) is a derivation and \( F : R \to R \) is a generalized derivation ([10]). \( iv) \) \( F([x, y]) \pm [x, y] \in Z(R) \) for all \( x, y \in I \), where \( R \) is a prime ring, \( I \) is a nonzero two sided ideal of \( R, (F, d) \) is a generalized derivation and \( d(Z(R)) \) is nonzero ([11]). \( v) \) \( F(xy) \in Z(R), F(xy) \pm yx \in Z(R), F(xy) \pm [x, y] \in Z(R) \) for all \( x, y \in I \), where \( R \) is a semiprime ring, \( I \) is a nonzero left ideal of \( R \) and \( (F, d) \) is a generalized derivation ([12]). \( vi) \) \( F(xy) \pm xy = 0, F(xy) \pm yx = 0 \), \( F(x)F(y) \pm xy = 0, F(x)F(y) \pm yx = 0, F(xy) \pm yx = 0, F(xy) \pm xy = 0 \), \( F(x)F(y) \pm [x, y] \in Z(R), F(x)F(y) \pm yx \in Z(R), F(x)F(y) \pm xy \in Z(R), F(x)F(y) \pm yx \in Z(R) \) for all \( x, y \in I \), where \( R \) is a semiprime ring, \( I \) is a nonzero left ideal of \( R \) and \( F \) is a multiplicative (generalized)-derivation ([11]).

Let \( R \) be a semiprime ring, \( F : R \to R \) be a multiplicative (generalized)-derivation associated with the map \( f \) and the map \( H : R \to R \) be a multiplicative left centralizer. In this paper, we study following conditions. \( i) \) \( F(xy) \pm H(xy) = 0 \), for all \( x, y \in R \). \( ii) \) \( F(xy) \pm H(xy) = 0 \), for all \( x, y \in R \). \( iii) \) \( F(xy) \pm H(xy) = 0 \), for all \( x, y \in R \). \( iv) \) \( F(xy) \pm H(xy) \in Z(R) \), for all \( x, y \in R \). \( v) \) \( F(xy) \pm H(xy) \in Z(R) \), for all \( x, y \in R \). \( vi) \) \( F(xy) \pm H(xy) \in Z(R) \), for all \( x, y \in R \). Moreover, given some corollaries for prime rings.

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2. Results

**Lemma 1.** [13, Lemma 3] Let $R$ be a prime ring and $d$ be a derivation of $R$ such that $[d(a), a] = 0$, for all $a \in R$. Then $R$ is commutative or $d$ is zero.

**Lemma 2.** Let $R$ be a semiprime ring. If $F$ is a multiplicative (generalized)-derivation associated with the map $f$, then $f$ is a multiplicative derivation, that is, $f(xy) = f(x)y + xf(y)$ for all $x, y \in R$.

*Proof.* Since $F$ is a multiplicative (generalized)-derivation we have

$$F(xyz) = F(xy)z + xF(yz), \forall x, y, z \in R$$

and

$$F((xy)z) = F(x)yz + xf(y)z + xyf(z), \forall x, y, z \in R.$$  

Hence we get,

$$xf(yz) = xf(y)z + xyf(z), \forall x, y, z \in R.$$  

From the last equation, we find that $R(f(yz) - f(y)z - yf(z)) = (0)$, for all $y, z \in R$. Since the semiprimeness of $R$, we have, $f(yz) = f(y)z + yf(z)$, for all $y, z \in R$. \hfill \Box

**Lemma 3.** Let $R$ be a semiprime ring and $F$ be a multiplicative (generalized)-derivation associated with $f$. If $F(xy) = 0$ holds for all $x, y \in R$, then $F = 0$.

*Proof.* By the assumption, we have

$$F(xy) = 0, \forall x, y \in R.$$  

If we replace $x$ by $xz$ with $z \in R$, we get

$$F(xyz) = 0, \forall x, y, z \in R.$$  

Since $F$ is a multiplicative (generalized)-derivation, we get

$$F(xz)y + zzf(y) = 0, \forall x, y, z \in R.$$  

Using the hypothesis we find that

$$zzf(y) = 0, \forall x, y, z \in R.$$  

Since $R$ is a semiprime ring, we obtain $xf(z) = 0$, for all $x, z \in R$. This means $f = 0$. From the definition of $F$, we get $F(xy) = F(x)y$, for all $x, y \in R$. By the hypothesis we see that

$$F(x)y = 0, \forall x, y \in R.$$  

From the semiprimeness of $R$, we find that $F = 0$. \hfill \Box

**Lemma 4.** Let $R$ be a semiprime ring and $F$ be a multiplicative (generalized)-derivation associated with $f$. If $F(xy) \in Z(R)$ holds for all $x, y \in R$, then $[f(x), x] = 0$ for all $x \in R$. 

**Proof.** By the hypothesis, we have

\[ F(xy) \in Z(R), \; \forall x, y \in R. \]

Taking \(yz\) instead of \(y\) with \(z \in R\), we get

\[ F(xyz) \in Z(R), \; \forall x, y, z \in R. \]

Since \(F\) is a multiplicative (generalized)-derivation, we have

\[ F(xy)z + xzf(z) \in Z(R), \; \forall x, y, z \in R. \]

From the hypothesis, we get

\[ [xyf(z), z] = 0, \; \forall x, y, z \in R. \]

Replacing \(x\) by \(rx\) with \(r \in R\), so we have

\[ [r, z]xyf(z) = 0, \; \forall x, y, z, r \in R. \]

In this equation replacing \(x\) by \(f(z)x\), we find that

\[ [r, z]f(z)xyf(z) = 0, \; \forall x, y, z, r \in R. \]

This implies that, for all \(x, y, s \in R\),

\[ [x, y]f(y)s[x, y]f(y) = [x, y]f(y)sxyf(y) - [x, y]f(y)sxf(y) = 0. \]

Since \(R\) is a semiprime ring, we find that

\[ [x, y]f(y) = 0, \; \forall x, y \in R. \]

Replacing \(x\) by \(xy\) with \(y \in R\), we have

\[ [x, y]yf(y) = 0, \; \forall x, y \in R. \]

Hence, we see that

\[ [x, y][f(y), y] = 0, \; \forall x, y \in R. \]

If we replace \(x\) by \(f(y)x\) and using the semiprimeness of \(R\), we get \([f(y), y] = 0\) for all \(y \in R\). \(\square\)

**Lemma 5.** Let \(R\) be a ring, \(F\) be a multiplicative (generalized)-derivation associated with \(f\) and \(H\) be a multiplicative left centralizer. If the map \(G : R \rightarrow R\) is defined as \(G(x) = F(x) \mp H(x)\) for all \(x \in R\), then \(G\) is a multiplicative (generalized)-derivation associated with \(f\).

**Proof.** We suppose that, for all \(x \in R\)

\[ G(x) = F(x) \mp H(x). \]

So we have, for all \(x, y \in R\)

\[
\begin{align*}
G(xy) &= F(xy) \mp H(xy) = F(x)y + xf(y) \mp H(x)y \\
&= (F(x) \mp H(x))y + xf(y) \\
&= G(x)y + xf(y).
\end{align*}
\]

Then \(G\) is a multiplicative (generalized)-derivation associated with \(f\). \(\square\)
Theorem 1. Let $R$ be a semiprime ring, $F : R \rightarrow R$ be a multiplicative (generalized)-derivation associated with $f$ and $H : R \rightarrow R$ be a multiplicative left centralizer. If $F(xy) + H(xy) = 0$ holds for all $x, y \in R$, then $f = 0$. Moreover, $F(xy) = F(x)y$ holds for all $x, y \in R$ and $F = \pm H$.

Proof. By the hypothesis, we have

$$F(xy) - H(xy) = 0, \forall x, y \in R.$$ 

So we have

$$G(xy) = 0, \forall x, y \in R$$

where $G(x) = F(x) - H(x)$. Using Lemma 3 and Lemma 5, we get

$$G = 0.$$ 

So we have

$$F = H.$$ (2.1)

Using the definition of $F$ and (2.1) in the hypothesis, we get

$$0 = F(xy) - H(xy) = F(x)y + xf(y) - H(x)y = xf(y), \forall x, y \in R.$$ 

Since $R$ is a semiprime ring, we obtain $f = 0$. Thus, we get $F(xy) = F(x)y$ for all $x, y \in R$. Similar proof shows that the same conclusion holds as $F(xy) + H(xy) = 0$, for all $x, y \in R$. In this case, we obtain $F = -H$. Therefore the proof is completed.

Theorem 2. Let $R$ be a semiprime ring, $F : R \rightarrow R$ be a multiplicative (generalized)-derivation associated with $f$ and $H : R \rightarrow R$ be a multiplicative left centralizer. If $F(xy) + H(yx) = 0$ holds for all $x, y \in R$, then $f = 0$. Moreover, $F(xy) = F(x)y$, for all $x, y \in R$ and $[F(x), x] = 0$, for all $x \in R$.

Proof. By the hypothesis, we have

$$F(xy) - H(yx) = 0, \forall x, y \in R.$$ (2.2)

Replacing $y$ by $yz$ with $z \in R$ in (2.2), we obtain

$$F(xyz) - H(yzx) = 0, \forall x, y, z \in R.$$ 

Since $F$ is a multiplicative (generalized)-derivation, we have

$$(F(xy) - H(yx))z + xyf(z) + H(y)[x, z] = 0, \forall x, y, z \in R.$$ 

Using (2.2) in the last equation, we get

$$xyf(z) + H(y)[x, z] = 0, \forall x, y, z \in R.$$ (2.3)

If we replace $z$ by $x$ in (2.3), we get

$$xyf(x) = 0, \forall x, y \in R.$$ 

Since $R$ is a semiprime ring, we obtain $xf(x) = f(x)x = 0$, for all $x \in R$. Hence we get,

$$[f(x), x] = 0, \forall x \in R.$$ (2.4)
If we replace $x$ by $xr$ with $r \in R$ in (2.3), we get the following equation. For all $x, y, z, r \in R$,

\[
0 = xrf(z) + H(y)[xr, z] \\
= xrf(z) + H(y)x[r, z] + H(y)[x, z]r + xyf(z)r - xyf(z)r \\
= xrf(z) + H(y)x[r, z] - xyf(z)r + (xyf(z) + H(y)[x, z])r.
\]

So, using (2.3) in this equation, we find that

\[
x[r, yf(z)] + H(y)x[r, z] = 0, \forall x, y, z, r \in R.
\]

In this equation replacing $r$ by $f(z)$ and using (2.4), we get

\[
x[f(z), y]f(z) = 0, \forall x, y, z \in R.
\]

Since $R$ is a semiprime ring, we have

\[
[f(z), y]f(z) = 0, \forall y, z \in R. \quad (2.5)
\]

Replacing $y$ by $yt$ with $t \in R$ in (2.5) and using (2.5), we find that

\[
[f(z), y]tf(z) = 0, \forall y, z, t \in R.
\]

This yields following equation.

\[
[f(z), y][f(z), y] = 0, \forall y, z, t \in R.
\]

From the semiprimeness of $R$, we find that

\[
[f(z), y] = 0, \forall y, z \in R. \quad (2.6)
\]

Replacing $x$ by $f(x)$ in (2.3) and using (2.6), we get, for all $x, y, z \in R$, $f(x)yf(z) = 0$. From the semiprimeness of $R$, this means

\[
f = 0. \quad (2.7)
\]

Hence, from the definition of $F$, we get

\[
F(xy) = F(x)y, \forall x, y \in R. \quad (2.8)
\]

Applying (2.7) to (2.3), we have

\[
H(y)[x, z] = 0, \forall x, y, z \in R.
\]

Replacing $y$ by $yz$ in the last equation and using respectively (2.2) and (2.8), we get

\[
F(z)y[x, z] = 0, \forall x, y, z \in R. \quad (2.9)
\]

If we replace $x$ by $F(z)$ in (2.9), we obtain

\[
F(z)y[F(z), z] = 0, \forall y, z \in R.
\]

Hence for $y, z \in R$, we get

\[
[F(z), z][F(z), z] = (F(z)z - zF(z))y[F(z), z] = 0.
\]
Consequently, since $R$ is a semiprime ring, we find that $[F(z), z] = 0$, for all $z \in R$. Similar proof shows that the same conclusion holds as $F(xy) + H(yx) = 0$, for all $x, y \in R$. Therefore the proof is completed. \qed

**Theorem 3.** Let $R$ be a semiprime ring, $F : R \to R$ be a multiplicative (generalized)-derivation associated with $f$ and $H : R \to R$ be a multiplicative left centralizer. If $F(x)F(y) + H(xy) = 0$ holds for all $x, y \in R$, then $f = 0$. Moreover, $F(xy) = F(x)y$ for all $x, y \in R$ and $[F(x), x] = 0$, for all $x \in R$.

**Proof.** By the hypothesis we have
\[
F(x)F(y) - H(xy) = 0, \quad \forall x, y \in R. \quad (2.10)
\]
Replacing $y$ by $yz$ with $z \in R$ in (2.10), we get
\[
F(x)F(yz) - H(xy)z = 0, \quad \forall x, y, z \in R.
\]
Since $F$ is a multiplicative (generalized)-derivation, we have
\[
(F(x)F(y) - H(xy))z + F(x)yz = 0, \quad \forall x, y, z \in R.
\]
Using (2.10) in the last equation, we get
\[
F(x)yz = 0, \quad \forall x, y, z \in R. \quad (2.11)
\]
Replacing $x$ by $ux$ with $u \in R$ in (2.11) and using (2.11), from the definition of $F$, we obtain
\[
uf(x)yz = 0, \quad \forall x, y, z, u \in R.
\]
In the last equation replacing $y$ by $yr, r \in R$ and using that $R$ is a semiprime ring, so we have $f = 0$. Thus, we get $F(xy) = F(x)y$ for all $x, y \in R$. In (2.10) replacing $x$ by $xy$, we have
\[
F(x)F(y) - H(xy)y = 0, \quad \forall x, y \in R. \quad (2.12)
\]
Multiplying (2.10) by $y$ on the right, we have
\[
F(x)F(y) - H(xy)y = 0, \quad \forall x, y \in R. \quad (2.13)
\]
Subtracting (2.12) from (2.13), we get
\[
F(x)[F(y), y] = 0, \quad \forall x, y \in R.
\]
Replacing $x$ by $xr$ with $r \in R$ in the last equation, we have
\[
F(x)r[F(y), y] = 0, \quad \forall x, y, r \in R.
\]
In this case, for $x, r \in R$, we find that
\[
[F(x), x]r[F(x), x] = (F(x)x - xF(x))r[F(x), x] = 0.
\]
Thus, since $R$ is a semiprime ring, we obtain $[F(x), x] = 0$, for all $x \in R$. Similar proof shows that the same conclusion holds as $F(x)F(y) + H(xy) = 0$, for all $x, y \in R$. \qed
Theorem 4. Let $R$ be a semiprime ring, $F : R \to R$ be a multiplicative (generalized)-derivation associated with $f$ and $H : R \to R$ be a multiplicative left centralizer. If $F(xy) + H(xy) \in Z(R)$ holds for all $x, y \in R$, then $[f(x), x] = 0$ for all $x \in R$.

Proof. By the supposition, we have
$$F(xy) + H(xy) \in Z(R), \forall x, y \in R.$$ So we have
$$G(xy) \in Z(R), \forall x, y \in R.$$ Using Lemma 4 and Lemma 5, we get
$$[f(x), x] = 0, \forall x \in R.$$ \hfill \Box

Theorem 5. Let $R$ be a semiprime ring, $F : R \to R$ be a multiplicative (generalized)-derivation associated with $f$ and $H : R \to R$ be a multiplicative left centralizer. If $F(xy) + H(yx) \in Z(R)$ holds for all $x, y \in R$, then $[f(x), x] = 0$ for all $x \in R$.

Proof. By the hypothesis, we have
$$F(xy) - H(yx) \in Z(R), \forall x, y \in R. \quad (2.14)$$
If we replace $y$ by $yz$ with $z \in R$ in $(2.14)$, we get
$$F(xyz) - H(yzx) \in Z(R), \forall x, y, z \in R.$$ Since $F$ is a multiplicative (generalized)-derivation, we find that
$$(F(xy) - H(yx))z + yxf(z) + H(y)[x, z] \in Z(R), \forall x, y, z \in R.$$ From the $(2.14)$, we have
$$[xyf(z), z] + [H(y)[x, z], z] = 0, \forall x, y, z \in R. \quad (2.15)$$
Replacing $x$ by $xz$ in $(2.15)$, we find that
$$[xzf(z), z] + [H(y)[x, z], z]z = 0, \forall x, y, z \in R. \quad (2.16)$$
Multiplying $(2.15)$ by $z$ on the right, we find that
$$[yf(z), z]z + [H(y)[x, z], z]z = 0, \forall x, y, z \in R. \quad (2.17)$$
Subtracting $(2.16)$ and $(2.17)$ side by side, so we get
$$[x][yf(z), z], z] = 0, \forall x, y, z \in R.$$ In the last equation, we replace $x$ by $rx$ with $r \in R$. Hence we get
$$[r, z][yf(z), z] = 0, \forall x, y, z, r \in R.$$ In this equation, replacing $r$ by $yf(z)$ and using that semiprimeness of $R$, we obtain
$$[yf(z), z] = 0, \forall y, z \in R.$$ If we take $f(z)y$ instead of $y$ and using last equation, we have
$$[f(z), z][yf(z)] = 0, \forall y, z \in R.$$ From the last equation we have,
$$[f(z), z][yf(z), z] = 0, \forall y, z \in R.$$ Since $R$ is a semiprime ring, we find that
$$[f(z), z] = 0, \forall z \in R.$$
Theorem 6. Let $R$ be a semiprime ring, $F : R \rightarrow R$ be a multiplicative (generalized)-derivation associated with $f$ and $H : R \rightarrow R$ be a multiplicative left centralizer. If $F(x)F(y) \neq H(xy) \in Z(R)$ holds for all $x, y \in R$, then $[f(x), x] = 0$ for all $x \in R$.

Proof. By the supposition, we have

$$F(x)F(y) - H(xy) \in Z(R), \quad \forall x, y \in R. \tag{2.18}$$

Replacing $y$ by $yz$ with $z \in R$ in (2.18), we get

$$F(x)F(yz) - H(xy)z \in Z(R), \quad \forall x, y, z \in R.$$ \hspace{1cm} (2.19)

Since $F$ is a multiplicative (generalized)-derivation, we have

$$(F(x)F(y) - H(xy))z + F(x)yz(z) \in Z(R), \quad \forall x, y, z \in R.$$ \hspace{1cm} (2.18)

Using (2.18), we get

$$[F(x)yz(z), z] = 0, \quad \forall x, y, z \in R. \tag{2.19}$$

Replacing $x$ by $xz$ in (2.19) and using (2.19), hence we have

$$[xf(z)yz(z), z] = 0, \quad \forall x, y, z \in R.$$ \hspace{1cm} (2.19)

In the last equation, replacing $x$ by $f(z)x$ and using this equation, we find that

$$[f(z), z]xf(z)yz(z) = 0, \quad \forall x, y, z \in R.$$ \hspace{1cm} (2.19)

This implies that

$$[f(z), z]xf(z)yz(z) = 0, \quad \forall x, y, z \in R.$$ \hspace{1cm} (2.19)

It gives that, $(R[f(z), z])^3 = 0$ for all $z \in R$. Since there is no nilpotent left ideal in semiprime rings ([2]), it gives that, $R[f(z), z] = 0$ for all $z \in R$. Hence using semiprimeness of $R$, we conclude that $[f(z), z] = 0$, for all $z \in R$. Similar proof shows that if $F(x)F(y) + H(xy) \in Z(R)$ holds for all $x, y \in R$, then $[f(x), x] = 0$ for all $x \in R$. \hfill \square

By Lemma 2, every multiplicative (generalized)-derivation $F : R \rightarrow R$ associated with an additive map $f$ is always a multiplicative generalized derivation in semiprime ring. Thus our next corollary is about multiplicative generalized derivation.

Corollary 1. Let $R$ be a prime ring and $F : R \rightarrow R$ be a multiplicative generalized derivation associated with a nonzero derivation $d$ and $H : R \rightarrow R$ be a multiplicative left centralizer. If one of the following conditions holds, for all $x, y \in R$, then $R$ is commutative.

i) $F(xy) \neq H(xy) \in Z(R)$,

ii) $F(xy) \neq H(yx) \in Z(R)$,

iii) $F(x)F(y) \neq H(xy) \in Z(R)$.
Proof. By Theorem 4, Theorem 5 and Theorem 6, we have \( d(x,x) = 0 \) for all \( x \in R \). Then by Lemma 1, \( R \) must be commutative.

Using the examples of similar in [1], the following examples show that the importance hypothesis of semiprimeness.

**Example 1.** Let \( R = \left\{ \left( \begin{array}{ccc} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \mid a, b, c \in \mathbb{Z} \right\} \), where \( \mathbb{Z} \) is the set of all integers and the maps \( F, f, H : R \rightarrow R \) defined by

\[
F \left( \begin{array}{ccc} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} 0 & 0 & \lambda b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad f \left( \begin{array}{ccc} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} 0 & 0 & \lambda b \lambda c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad H \left( \begin{array}{ccc} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} 0 & \lambda a & \lambda b \\ 0 & 0 & \lambda c \\ 0 & 0 & 0 \end{array} \right), \quad \text{where } \lambda \in \mathbb{Z}.
\]

Since \( \left( \begin{array}{ccc} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) R \left( \begin{array}{ccc} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) = (0) \), \( R \) is not semiprime. Moreover, it is easy to show that, \( F \) is a multiplicative (generalized)-derivation associated with \( f \) and \( H(xy) = H(x)y, F(xy) - H(xy) = 0 \) holds for all \( x, y \in R \). But, we observe that \( f(R) \neq 0 \) and \( F(xy) \neq F(x)y \) for \( x, y \in R \). Hence the semiprimeness hypothesis in the Theorem 1 is crucial.

**Example 2.** Let \( R = \left\{ \left( \begin{array}{ccc} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \mid a, b, c \in \mathbb{Z} \right\} \), where \( \mathbb{Z} \) is the set of all integers and the maps \( F, f, H : R \rightarrow R \) defined by

\[
F \left( \begin{array}{ccc} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} 0 & \lambda a & 0 \\ 0 & 0 & \lambda c \\ 0 & 0 & 0 \end{array} \right), \quad f \left( \begin{array}{ccc} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} 0 & \lambda ab & \lambda b^2 \\ 0 & 0 & -\lambda c \\ 0 & 0 & 0 \end{array} \right), \quad H \left( \begin{array}{ccc} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) = \left( \begin{array}{ccc} 0 & \lambda^2 a & \lambda^2 b \\ 0 & 0 & \lambda^2 c \\ 0 & 0 & 0 \end{array} \right), \quad \text{where } \lambda \in \mathbb{Z}.
\]

Then \( R \) is not semiprime and it is easy to show that, \( F \) is a multiplicative (generalized)-derivation associated with \( f \) and \( H(xy) = H(x)y, F(xy) - H(xy) = 0 \) holds for all \( x, y \in R \). But, we observe that \( f(R) \neq 0 \) and \( F(xy) \neq F(x)y \) for \( x, y \in R \). Hence the semiprimeness hypothesis in the Theorem 3 is essential.
Example 3. Let $R = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the set of all integers and the maps $F, f, H : R \rightarrow R$ defined by

$$F \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a^2 & 0 & 0 \\ b+c & 0 & 0 \end{pmatrix}, \quad f \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b^2 & 0 & 0 \end{pmatrix}$$

$$H \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ ab & 0 & 0 \end{pmatrix}.$$

Since $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} R \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = (0)$, $R$ is not a semiprime ring. It yields that $F$ is a multiplicative (generalized)-derivation associated with $f$ and $H(xy) = H(x)y, F(x)F(y) - H(xy) = 0$ holds for all $x, y \in R$. But, we see that $f(R) \neq 0$ and $F(xy) \neq F(x)y$ for $x, y \in R$. Hence the semiprimeness hypothesis in the Theorem 3 is essential.

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