VECTOR-VALUED CESÀRO SUMMABLE GENERALIZED LORENTZ SEQUENCE SPACE

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ABSTRACT. The main purpose of this paper is to introduce Cesàro summable generalized Lorentz sequence space \( C_1[d(v, p)] \). We study some topological properties of this space and obtain some inclusion relations.

1. Introduction

Throughout this work, \( \mathbb{N} \), \( \mathbb{R} \) and \( \mathbb{C} \) denote the set of positive integers, real numbers and complex numbers, respectively. For some properties of sequences, we refer to [4, 8].

For \( 1 \leq p < \infty \), the Cesàro sequence space is defined by

\[
C_{\text{es}p} = \left\{ x \in w : \sum_{j=1}^{\infty} \left( \frac{1}{j} \sum_{i=1}^{j} |x(i)| \right)^p < \infty \right\},
\]

equipped with norm

\[
\|x\| = \left( \sum_{j=1}^{\infty} \left( \frac{1}{j} \sum_{i=1}^{j} |x(i)| \right)^p \right)^{\frac{1}{p}}.
\]

This space was first introduced by Shiue [14]. It is very useful in the theory of matrix operators and others. Later, many authors studied this space [see 1, 5, 11, 13].

Let \( (E, \|\|) \) be a Banach space. The Lorentz sequence space \( l(p, q, E) \) (or \( l_{p,q}(E) \)) for \( 1 \leq p, q \leq \infty \) is the collection of all sequences \( \{a_i\} \in c_0(E) \) such that

\[
\|\{a_i\}\|_{p,q} = \left\{ \begin{array}{ll}
\left( \sum_{i=1}^{\infty} \left( \frac{i^{q/p-1}}{i^{1/p}} \|a_{\phi(i)}\|^q \right)^{1/q} \right) & \text{for } 1 \leq p < \infty, \; 1 \leq q < \infty \\
\sup_i i^{q/p} \|a_{\phi(i)}\| & \text{for } 1 \leq p \leq \infty, \; q = \infty
\end{array} \right.
\]

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is finite, where \( \{ \| a_{\phi(i)} \| \} \) is non-increasing rearrangement of \( \{ \| a_i \| \} \) (We can interpret that the decreasing rearrangement \( \{ \| a_{\phi(i)} \| \} \) is obtained by rearranging \( \{ \| a_i \| \} \) in decreasing order). This space was introduced by Miyazaki in [9] and examined comprehensively by Kato in [3] (see also [6, 7]).

A weight sequence \( v = \{ v(i) \} \) is a positive decreasing sequence such that \( v(1) = 1, \lim_{i \to \infty} v(i) = 0 \) and \( \lim_{i \to \infty} V(i) = \infty \), where \( V(i) = \sum_{n=1}^{i} v(n) \) for every \( i \in \mathbb{N} \).

Popa [12] defined the generalized Lorentz sequence space \( d(v, p) \) for \( 0 < p < \infty \) as follows

\[
d(v, p) = \left\{ x = \{ x_i \} \in w : \| x \|_{v, p} = \sup_{\pi} \left( \sum_{i=1}^{\infty} |x_{\pi(i)}|^p v(i) \right)^{1/p} < \infty \right\},
\]

where \( \pi \) ranges over all permutations of the positive integers and \( v = \{ v(i) \} \) is a weight sequence. It is know that \( d(v, p) \subset c_0 \) and hence for each \( x \in d(v, p) \) there exists a non-increasing rearrangement \( \{ x^* \} = \{ x_i^* \} \) of \( x \) and

\[
\| x \|_{v, p} = \left( \sum_{n=1}^{\infty} |x_i^*|^p v(i) \right)^{\frac{1}{p}}
\]

(see [10, 12]).

Let \( (X, \| \cdot \|) \) be a Banach space and \( v = \{ v(k) \} \) be a weight sequence. We introduce the vector-valued Cesáro summable generalized Lorentz sequence space \( C_1 [d(v, p)] \) for \( 0 < p < \infty \). The space \( C_1 [d(v, p)] \) is the collection of all \( X \)-valued 0-sequences \( \{ x_n \} \) \( \{ x_n \} \) \( \{ x_n \} \) \( \{ x_n \} \) such that

\[
\left( \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \| x_{\phi(n)} \| \right]^p v(k) \right)^{\frac{1}{p}}
\]

is finite, where \( \{ \| x_{\phi(n)} \| \} \) is non-increasing rearrangement of \( \{ \| x_n \| \} \).

We shall need the following lemmas.

**Lemma 1.** (Hardy, Littlewood and Pólya [2]) Let \( \{ a_i \}_{1 \leq i \leq n} \) and \( \{ b_i \}_{1 \leq i \leq n} \) be two sequences of positive numbers. Then we have

\[
\sum_{i} a_i^* \cdot b_i \leq \sum_{i} a_i \cdot b_i \leq \sum_{i} a_i^+ \cdot b_i^+;
\]

where \( \{ a_i^* \} \) is the non-increasing rearrangements of sequence \( \{ a_i \}_{1 \leq i \leq n} \) and \( \{ b_i^+ \} \) and \( \{ b_i \} \) are the non-increasing and non-decreasing rearrangements of sequence \( \{ b_i \}_{1 \leq i \leq n} \), respectively.

**Lemma 2.** (Kato [3]) Let \( \{ x_i^{(n)} \} \) be an \( X \)-valued double sequence such that \( \lim_{i \to \infty} x_i^{(n)} = 0 \) for each \( \mu \in \mathbb{N} \) and let \( \{ x_i \} \) be an \( X \)-valued sequence such that
lim_{\mu \to \infty} x_i^{(\mu)} = x_i \text{ (uniformly in } i\text{). Then } \lim_{i \to \infty} x_i = 0 \text{ and for each } i \in \mathbb{N}

\|x_\phi(i)\| \leq \lim_{\mu \to \infty} \|x_i^{(\mu)}\|,

where \{\|x_\phi(i)\|\} and \{\|x_\phi^{(\mu)}(i)\|\}_i are the non-increasing rearrangements of \{\|x_i\|\} and \{\|x_i^{(\mu)}\|\}_i, respectively.

2. MAIN RESULTS

**Theorem 1.** The space $C_1 [d(v,p)]$ for $0 < p < \infty$ is a linear space over the field $K = \mathbb{R}$ or $\mathbb{C}$.

**Proof.** Let $x, y \in C_1 [d(v,p)]$. Since $v$ is non-increasing, the non-increasing rearrangements of $v$ is itself. Thus, using the inequality $\sum_i a_i \cdot b_i \leq \sum_i a_i^* \cdot b_i^*$ from Lemma 1, we have

$$\sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \|x_\phi(n) + y_\psi(n)\| \right]^p v(k) \leq \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} (\|x_\phi(n)\| + \|y_\psi(n)\|) \right]^p v(k)$$

$$\leq D \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \|x_\phi(n)\| \right]^p v(k) + D \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \|y_\psi(n)\| \right]^p v(k)$$

$$\leq D \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \|x_\phi(n)\| \right]^p v(k) + D \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \|y_\psi(n)\| \right]^p v(k)$$

$$< \infty,$$

where $D = \max \{1, 2^{p-1}\}$. Here \{\|x_\phi(n)\|\}, \{\|y_\psi(n)\|\} and \{\|x_\phi(n) + y_\psi(n)\|\} denote the non-increasing rearrangements of the sequences \{\|x_i\|\}, \{\|y_i\|\} and \{\|x_i + y_i\|\}, respectively. Let $\alpha \in K$. Hence we get

$$\sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \|\alpha x_\phi(n)\| \right]^p v(k) = \sum_{k=1}^{\infty} \left[ \frac{|\alpha|}{k} \sum_{n=1}^{k} \|x_\phi(n)\| \right]^p v(k)$$

$$= |\alpha|^p \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \|x_\phi(n)\| \right]^p v(k)$$

$$< \infty.$$
This shows that $x + y \in C_1[\{d(v, p)\} \cup \alpha x \in C_1[\{d(v, p)\}]$ and so $C_1[\{d(v, p)\}]$ is a linear space.

**Theorem 2.** The space $C_1[\{d(v, p)\}]$ for $1 \leq p < \infty$ is normed space with the norm

$$\|x\|_{C,v,p} = \left( \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \|x_{\phi(n)}\| \right]^p \right)^{\frac{1}{p}},$$

where $\{\|x_{\phi(n)}\|\}$ denotes the non-increasing rearrangements of $\{\|x_n\|\}$.

**Proof.** It is clear that $\|0\|_{C,v,p} = 0$. Let $\|x\|_{C,v,p} = 0$. Then we have $\frac{1}{k} \sum_{n=1}^{k} \|x_{\phi(n)}\| = 0$ for all $k \in \mathbb{N}$. Hence we get $\|x_{\phi(n)}\| = 0$ for all $n \in \mathbb{N}$ and so $x = 0$.

Let $x, y \in C_1[\{d(v, p)\}]$. Since weight sequence $v$ is decreasing, the non-increasing rearrangements of $v$ is itself. Thus, using the inequality $\sum_i a_i \cdot b_i \leq \sum_i a_i^* \cdot b_i^*$ from Lemma 1, we have

$$\|x + y\|_{C,v,p} = \left( \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \|x_{\psi(n)} + y_{\psi(n)}\| \right]^p \right)^{\frac{1}{p}} \leq \left( \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \|x_{\psi(n)}\| \right]^p \right)^{\frac{1}{p}} + \left( \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \|y_{\psi(n)}\| \right]^p \right)^{\frac{1}{p}} \leq \|x\|_{C,v,p} + \|y\|_{C,v,p},$$

where $\{\|x_{\phi(n)}\|\}, \{\|y_{\phi(n)}\|\}$ and $\{\|x_{\psi(n)} + y_{\psi(n)}\|\}$ denote the non-increasing rearrangements of $\{\|x_n\|\}, \{\|y_n\|\}$ and $\{\|x_n + y_n\|\}$, respectively.

Let $\lambda$ be an element of $K$ and let $x$ be a vector in $C_1[\{d(v, p)\}]$. Hence we have

$$\|\lambda x\|_{C,v,p} = \left( \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \|\lambda x_{\phi(n)}\| \right]^p \right)^{\frac{1}{p}} = |\lambda| \left( \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \|x_{\phi(n)}\| \right]^p \right)^{\frac{1}{p}} = |\lambda| \|x\|_{C,v,p}.$$  

**Theorem 3.** The space $C_1[\{d(v, p)\}]$ for $1 \leq p < \infty$ is complete with respect to its norm.
Proof. Let \( \{x^{(s)}\} \) be an arbitrary Cauchy sequence in \( C_1 [d(v,p)] \) with \( x^{(s)} = \{x^{(s)}_n\}_{n=1}^{\infty} \) for all \( s \in \mathbb{N} \). Then we have

\[
\lim_{s,t \to \infty} \left\| x^{(s)} - x^{(t)} \right\|_{C_1 [d(v,p)]} = \lim_{s,t \to \infty} \left( \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \left\| x^{(s)}_{\pi_{s,t}(n)} - x^{(t)}_{\pi_{s,t}(n)} \right\|_v \right]^p \right)^{\frac{1}{p}} = 0,
\]

where \( \{ \left\| x^{(s)}_{\pi_{s,t}(n)} - x^{(t)}_{\pi_{s,t}(n)} \right\|_v \} \) denotes the non-increasing rearrangement of \( \left\{ \left\| x^{(s)} - x^{(t)} \right\|_v \right\} \). Hence we obtain \( \lim_{s,t \to \infty} \left\| x^{(s)}_{\pi_{s,t}(n)} - x^{(t)}_{\pi_{s,t}(n)} \right\|_v = 0 \) for each \( n \in \mathbb{N} \)
and so \( \{x^{(s)}_n\} \), for a fixed \( n \in \mathbb{N} \), is a Cauchy sequence in \( X \).

Then, there exists \( x_n \in X \) such that \( x^{(s)}_n \to x_n \) as \( s \to \infty \). Let \( x = \{x_n\} \).
Since \( \lim_{n \to \infty} x^{(s)}_n = 0 \) for each \( s \in \mathbb{N} \), by Lemma 2 we have \( \lim_{n \to \infty} x_n = 0 \).
Therefore we can choose the non-increasing rearrangement \( \left\{ \left\| x^{(s)}_{\pi_{s,t}(n)} - x^{(t)}_{\pi_{s,t}(n)} \right\|_v \right\}_{n} \)
of \( \left\{ \left\| x_n - x^{(t)}_n \right\|_v \right\} \). Also, for an arbitrary \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that

\[
\left( \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \left\| x^{(s)}_{\pi_{s,t}(n)} - x^{(t)}_{\pi_{s,t}(n)} \right\|_v \right]^p \right)^{\frac{1}{p}} < \varepsilon
\]

for \( s, t > N \). Let \( t \) be an arbitrary positive integer with \( t > N \) and fixed. If we put

\[
y^{(s)}_n = x^{(s)}_n - x^{(t)}_n \quad \text{and} \quad y_n = x_n - x^{(t)}_n,
\]
then we have

\[
\lim_{n \to \infty} \ y^{(s)}_n = 0 \quad \text{for each} \ s \in \mathbb{N} \quad \text{and} \quad \lim_{n \to \infty} \ y^{(s)}_n = y_n \quad \text{(uniformly in} \ n).\]

Thus by Lemma 2 we get

\[
\left\| y^{(s)}_n \right\|_v \leq \lim_{n \to \infty} \left\| y^{(s)}_{\pi_{s,t}(n)} \right\|_v
\]
for each \( n \in \mathbb{N} \), that is,

\[
\left\| x^{(s)}_{\pi_{s,t}(n)} - x^{(t)}_{\pi_{s,t}(n)} \right\|_v \leq \lim_{n \to \infty} \left\| x^{(s)}_{\pi_{s,t}(n)} - x^{(t)}_{\pi_{s,t}(n)} \right\|.
\]
for each \( n \in \mathbb{N} \). Hence, by (2), (3) we get

\[
\left\| x - x^{(t)} \right\|_{C,v,p} = \left( \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{k} \left\| x_{\pi_t(n)} - x^{(t)}_{\pi_t(n)} \right\|^p v(k) \right)^{1/p} \\
\leq \left( \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{k} \lim_{s \to 1} \left\| x^{(s)}_{\pi_s,t(n)} - x^{(t)}_{\pi_s,t(n)} \right\|^p v(k) \right)^{1/p} \\
= \lim_{s \to 1} \left( \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{k} \left\| x^{(s)}_{\pi_s,t(n)} - x^{(t)}_{\pi_s,t(n)} \right\|^p v(k) \right)^{1/p} \\
< \varepsilon.
\]

Also, since \( C_1 [d(v,p)] \) is a linear space we have \( \{ x_n \} = \{ x_n - x^{(N)}_n \} \in C_1 [d(v,p)] \). Hence the space \( C_1 [d(v,p)] \) is complete with respect to its norm.

**Theorem 4.** Let \( 1 < p < \infty \). Then, the inclusion \( d(v,p) \subset C_1 [d(v,p)] \) holds.

**Proof.** Let \( x \in d(v,p) \). Then there exists \( T > 0 \) such that

\[
\lim_{m \to \infty} \left( \sum_{n=1}^{m} \left\| x_{\phi(n)} \right\|^p v(n) \right)^{1/p} = \left( \sum_{n=1}^{\infty} \left\| x_{\phi(n)} \right\|^p v(n) \right)^{1/p} < T < \infty,
\]

where \( \{ \left\| x_{\phi(n)} \right\| \} \) denotes the non-increasing rearrangements of \( \{ \left\| x_n \right\| \} \). Since \( \sum_{k=1}^{\infty} \frac{1}{k^p} < \infty \) for \( 1 < p < \infty \) and \( v \) is decreasing, we get

\[
\sum_{k=1}^{\infty} \frac{1}{k^p} \sum_{n=1}^{k} \left\| x_{\phi(n)} \right\|^p v(k) = \sum_{k=1}^{\infty} \frac{1}{k^p} \left( \sum_{n=1}^{k} \left\| x_{\phi(n)} \right\|^p v(n) \right) \leq \max \{ 1, 2^{p-1} \} \sum_{k=1}^{\infty} \frac{1}{k^p} \left( \sum_{n=1}^{k} \left\| x_{\phi(n)} \right\|^p v(n) \right) < \infty.
\]

This completes the proof.

**Theorem 5.** If \( 1 \leq p < q < \infty \), then \( C_1 [d(v,p)] \subset C_1 [d(v,q)] \).
Proof. Let \( x \in C_1 [d(v, p)] \) and let \( \{ \| x_{\phi(n)} \| \} \) denotes the non-increasing rearrangement of \( \{ \| x_n \| \} \). Since \( v(k) \) is decreasing we have
\[
\left( \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \| x_{\phi(n)} \| \right]^p v(k) \right)^{\frac{1}{p}} \geq \left( \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{m} \| x_{\phi(n)} \| \right]^p v(k) \right)^{\frac{1}{p}}
\]
\[
\geq \left( \sum_{k=1}^{m} \| x_{\phi(n)} \| \right)^{\frac{1}{p}} v(k)
\]
for every \( m \in \mathbb{N} \). Hence we get
\[
\| x_{\phi(m)} \| \leq (v(m))^{-\frac{1}{p}} m^{-\frac{1}{q}} \left( \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \| x_{\phi(n)} \| \right]^p v(k) \right)^{\frac{1}{p}}
\]
for every \( m \in \mathbb{N} \). Thus
\[
\sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \| x_{\phi(n)} \| \right]^q v(k) = \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \| x_{\phi(n)} \| \right]^{q-p} \left[ \frac{1}{k} \sum_{n=1}^{k} \| x_{\phi(n)} \| \right]^p v(k)
\]
\[
\leq \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} (v(n))^{-\frac{1}{p}} \| x \|_{C_v, p} \right] \left[ \frac{1}{k} \sum_{n=1}^{k} \| x_{\phi(n)} \| \right]^{p} v(k)
\]
\[
\leq \left( (v(n))^{-\frac{1}{p}} \| x \|_{C_v, p} \right) ^{q-p} \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \| x_{\phi(n)} \| \right]^{p} v(k)
\]
\[
< \infty.
\]
This implies that \( x \in C_1 [d(v, q)] \). \( \square \)

Comment. If we put \( \Delta^m x \) instead of \( x \), where \( m \in \mathbb{N} \) and \( \Delta^0 x_k = \{ x_k \} \), \( \Delta x_k = x_k - x_{k+1} \), \( \Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1} = \sum_{v=1}^{m} (-1)^v \binom{m}{v} x_{k+v} \) for all \( k \in \mathbb{N} \) in the definition of \( C_1 [d(v, p)] \), we obtain Cesàro summable generalized Lorentz difference sequence space \( C_1 [d(v, \Delta^m, p)] \) of order \( m \). It can be shown that the sequence space \( C_1 [d(v, \Delta^m, p)] \) is a Banach space with norm
\[
\| x \|_{C_v, \Delta^m, p} = \sum_{k=1}^{m} \| x_{\phi(k)} \| + \left( \sum_{k=1}^{\infty} \left[ \frac{1}{k} \sum_{n=1}^{k} \| \Delta^m x_{\phi(n)} \| \right]^{p} v(k) \right)^{\frac{1}{p}},
\]
where \( \{ \| \Delta^m x_{\phi(n)} \| \} \) denotes the non-increasing rearrangements of \( \{ \| \Delta^m x_n \| \} \), and properties in this work.
References


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