CURVES OF CONSTANT BREADTH ACCORDING TO TYPE-2 BISHOP FRAME IN $E^3$

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Abstract. In this paper, we study the curves of constant breadth according to type-2 Bishop frame in the 3-dimensional Euclidean Space $E^3$. Moreover some characterizations of these curves are obtained.

1. Introduction

In 1780, L. Euler studied curves of constant breadth in the plane [3]. Thereafter, this issue investigated by many geometers [2, 4, 12]. Constant breadth curves are an important subject for engineering sciences, especially, in cam designs [17]. M. Fujiwara introduced constant breadth for space curves and surfaces [4]. D. J. Struik published some important publications on this subject [16]. O. Kose expressed some characterizations for space curves of constant breadth in Euclidean 3-space [10] and M. Sezer researched space curves of constant breadth and obtained a criterion for these curves [15]. A. Magden and O. Kose obtained constant breadth curves in Euclidean 4-space [11]. Characterizations for spacelike curves of constant breadth in Minkowski 4-space were given by M. Kazaz et al. [9]. S. Yilmaz and M. Turgut studied partially null curves of constant breadth in semi-Riemannian space [18]. The properties of these curves in 3-dimensional Galilean space were given by D. W. Yoon [20]. H. Gun Bozok and H. Oztekin investigated an explicit characterization of mentioned curves according to Bishop frame in 3-dimensional Euclidean space [5]. The curve of constant breadth on the sphere studied by W. Blaschke [2]. Furthermore, the method related to the curves of constant breadth for the kinematics of machinery was given by F. Reuleaux [14].

L. R. Bishop defined Bishop frame, which is known alternative or parallel frame of the curves with the help of parallel vector fields [1]. Then, S. Yılmaz and M. Turgut examined a new version of the Bishop frame which is called type-2 Bishop frame [19]. Thereafter, E. Ozyılmaz studied classical differential geometry of curves according to type-2 Bishop trihedra [13].
In this paper, we used the theory of the curves with respect to type-2 Bishop frame. Then, we gave some characterizations for curves of constant breadth according to type-2 Bishop frame.

2. Preliminaries

The standard flat metric of 3-dimensional Euclidean space \( E^3 \) is given by

\[
(\cdot, \cdot) : dx_1^2 + dx_2^2 + dx_3^2
\]  

(2.1)

where \((x_1, x_2, x_3)\) is a rectangular coordinate system of \( E^3 \). For an arbitrary vector \( x \) in \( E^3 \), the norm of this vector is defined by \( \|x\| = \sqrt{(x, x)} \). \( \alpha \) is called a unit speed curve, if \( (\alpha', \alpha') = 1 \). Suppose that \( \{t, n, b\} \) is the moving Frenet–Serret frame along the curve \( \alpha \) in \( E^3 \). For the curve \( \alpha \), the Frenet-Serret formulae can be given as

\[
\begin{align*}
t' &= \kappa n \\
n' &= -\kappa t + \tau b \\
b' &= -\tau n
\end{align*}
\]  

(2.2)

where

\[
\begin{align*}
\langle t, t \rangle &= \langle n, n \rangle = \langle b, b \rangle = 1, \\
\langle t, n \rangle &= \langle t, b \rangle = \langle n, b \rangle = 0.
\end{align*}
\]

and here, \( \kappa = \kappa(s) = \|t'(s)\| \) and \( \tau = \tau(s) = -\langle n, b' \rangle \). Furthermore, the torsion of the curve \( \alpha \) can be given

\[
\tau = \frac{|\alpha', \alpha'', \alpha'''|}{\kappa^2}.
\]

Along the paper, we assume that \( \kappa \neq 0 \) and \( \tau \neq 0 \).

Bishop frame is an alternative approachment to define a moving frame. Assume that \( \alpha(s) \) is a unit speed regular curve in \( E^3 \). The type-2 Bishop frame of the \( \alpha(s) \) is expressed as [19]

\[
\begin{align*}
N'_1 &= -k_1 B, \\
N'_2 &= -k_2 B, \\
B' &= k_1 N_1 + k_2 N_2.
\end{align*}
\]  

(2.3)

The relation matrix may be expressed as

\[
\begin{bmatrix}
t \\
n \\
b
\end{bmatrix} = \begin{bmatrix}
\sin \theta(s) & -\cos \theta(s) & 0 \\
\cos \theta(s) & \sin \theta(s) & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
N_1 \\
N_2 \\
B
\end{bmatrix}.
\]  

(2.4)
where $\theta(s) = \int_0^s \kappa(s) \, ds$. Then, type-2 Bishop curvatures can be defined in the following

$$k_1(s) = -\tau(s) \cos \theta(s),$$
$$k_2(s) = -\tau(s) \sin \theta(s).$$

On the other hand,

$$\theta' = \kappa = \frac{(k_2/k_1)'}{1 + (k_2/k_1)^2}.$$

The frame $\{N_1, N_2, B\}$ is properly oriented, $\tau$ and $\theta(s) = \int_0^s \kappa(s) \, ds$ are polar coordinates for the curve $\alpha$. Then, $\{N_1, N_2, B\}$ is called type-2 Bishop trihedra and $k_1, k_2$ are called Bishop curvatures.

The characterizations of inclined curves in $E^n$ is given [7] and [8] as follows

**Theorem 1.** $\alpha$ is an inclined curve in $E^n$ $\iff$ $\sum_{i=1}^{n-2} H_i^2 = \text{const}$ and $\alpha$ is an inclined curve in $E^{n-1} \iff \det (V'_1, V'_2, ..., V'_n) = 0$.

**Theorem 2.** Let $M \subset E^3$ is a curve given by $(I, \alpha)$ chart. Then $M$ is an inclined curve if and only if $H(s) = \frac{k_1(s)}{k_2(s)}$ is constant for all $s \in I$.

3. **Curves of Constant Breadth According to type-2 Bishop Frame in $E^3$**

Let $X = \bar{X}(s)$ be a simple closed curve in $E^3$. These curves will be denoted by $(C)$. The normal plane at every point $P$ on the curve meets the curve at a single point $Q$ other than $P$. The point $Q$ is called the opposite point of $P$. Considering a curve $\alpha$ which have parallel tangents $\bar{T}$ and $\bar{T}^*$ in opposite points $X$ and $X^*$ of the curve as in [4]. A simple closed curve of constant breadth which have parallel tangents in opposite directions can be introduced by

$$X^*(s) = X(s) + m_1(s) N_1 + m_2(s) N_2 + m_3(s) B \quad (3.1)$$

where $X$ and $X^*$ are opposite points and $N_1, N_2, B$ denote the type-2 Bishop frame in $E^3$ space. If $N_1$ is taken instead of tangent vector and differentiating equation (3.1) we have

$$\frac{dX^*}{ds} = \frac{dX^*}{ds^*} \frac{ds^*}{ds} = N_1 \frac{ds^*}{ds} = \left(1 + \frac{dm_1}{ds} + m_3 k_1 \right) N_1 + \left( \frac{dm_2}{ds} + m_3 k_2 \right) N_2 + \left( \frac{dm_3}{ds} - m_1 k_1 - m_2 k_2 \right) B \quad (3.2)$$
where \( k_1 \) and \( k_2 \) are the first and the second curvatures of the curve, respectively [6]. Since \( N_1^* = -N_1 \), we obtain
\[
\frac{ds^*}{ds} + \frac{dm_1}{ds} + m_3 k_1 + 1 = 0,
\]
\[
\frac{dm_2}{ds} + m_3 k_2 = 0,
\]
\[
\frac{dm_3}{ds} - m_1 k_1 - m_2 k_2 = 0.
\]
(3.3)

Suppose that \( \phi \) is the angle between the tangent of the curve \((C)\) at point \( X(s) \) with a given fixed direction and \( \frac{d\phi}{ds} = k_1 \), then the equation (3.3) can be written as
\[
\frac{dm_1}{d\phi} = -m_3 - f(\phi),
\]
\[
\frac{dm_2}{d\phi} = -\rho k_2 m_3,
\]
\[
\frac{dm_3}{d\phi} = m_1 + \rho k_2 m_2,
\]
(3.4)

where \( f(\phi) = \rho + \rho^* \), \( \rho = \frac{1}{k_1} \) and \( \rho^* = \frac{1}{k_1^*} \) denote the radius of curvatures at \( X \) and \( X^* \), respectively. If we consider equation (3.4), we get
\[
\frac{k_1}{k_2} m_1'' + \left( \frac{k_1}{k_2} \right)' m_1'' + \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) m_1' + \left( \frac{k_1}{k_2} \right)' m_1 + \left( \frac{k_1}{k_2} \right) f(\phi)'' + \left( \frac{k_1}{k_2} \right)' f(\phi)' + \left( \frac{k_2}{k_1} \right) f(\phi) = 0
\]
(3.5)

This equation is a characterization for \( X^* \). If the distance between the opposite points of \((C)\) and \((C^*)\) is constant, then
\[
||X^* - X||^2 = m_1^2 + m_2^2 + m_3^2 = l^2, \; l \in \mathbb{R}.
\]

Hence, we write
\[
m_1 \frac{dm_1}{d\phi} + m_2 \frac{dm_2}{d\phi} + m_3 \frac{dm_3}{d\phi} = 0
\]
(3.6)

By considering system (3.4), we obtain
\[
m_1 \left( \frac{dm_1}{d\phi} + m_3 \right) = 0.
\]
(3.7)

Thus we can write \( m_1 = 0 \) or \( \frac{dm_1}{d\phi} = -m_3 \). Then, we consider these situations with some subcases.
**Case 1.** If \( \frac{dm_1}{d\phi} = -m_3 \), then \( f(\phi) = 0 \). So, \((C^*)\) is translated by the constant vector

\[
u = m_1 N_1 + m_2 N_2 + m_3 B
\]  

(3.8)

of \((C)\). Here, let us solve the equation (3.5), in some special cases.

**Case 1.1** Let \( X \) be an inclined curve. Then the equation (3.5) can be written as follows,

\[
\frac{d^3m_1}{d\phi^3} + \left( 1 + \frac{k_2^2}{k_1^2} \right) \frac{dm_1}{d\phi} = 0.
\]  

(3.9)

The general solution of this equation is

\[
m_1 = c_1 + c_2 \cos \sqrt{1 + \frac{k_2^2}{k_1^2} \phi} + c_3 \sin \sqrt{1 + \frac{k_2^2}{k_1^2} \phi}
\]  

(3.10)

And therefore, we have \( m_2 \) and \( m_3 \), respectively,

\[
m_2 = k_2 \left( c_2 \cos \sqrt{1 + \frac{k_2^2}{k_1^2} \phi} \right) + k_2 \left( c_3 \sin \sqrt{1 + \frac{k_2^2}{k_1^2} \phi} \right)
\]  

(3.11)

\[
m_3 = c_2 \sqrt{1 + \frac{k_2^2}{k_1^2} \phi} - c_3 \sqrt{1 + \frac{k_2^2}{k_1^2} \phi}
\]  

(3.12)

where \( c_1 \) and \( c_2 \) are real numbers.

**Corollary 1.** Position vector of \( X^* \) can be formed by the equations (3.10), (3.11) and (3.12). Also the curvature of \( X^* \) is obtained as

\[
k_1^* = -k_1.
\]  

(3.13)

**Case 2.** \( m_1 = 0 \). Then, considering equation (3.5) we get

\[
\left( \frac{k_1}{k_2} \right) f(\phi)'' + \left( \frac{k_1}{k_2} \right) f(\phi)' + \left( \frac{k_2}{k_1} \right) f(\phi) = 0
\]  

(3.14)

**Case 2.1** Suppose that \( X \) is an inclined curve. The equation (3.14) can be rewrite as

\[
f(\phi)'' + \left( \frac{k_2}{k_1} \right)^2 f(\phi) = 0.
\]  

(3.15)

So, the solution of above differential equation is

\[
f(\phi) = L_1 \cos \frac{k_2}{k_1} \phi + L_2 \sin \frac{k_2}{k_1} \phi
\]  

(3.16)
where $L_1$ and $L_2$ are real numbers. Using above equation we obtain

\[ m_2 = L_1 \sin \frac{k_2}{k_1} \phi - L_2 \cos \frac{k_2}{k_1} \phi \]  
\[ m_3 = -L_1 \cos \frac{k_2}{k_1} \phi - L_2 \sin \frac{k_2}{k_1} \phi = -\rho - \rho^* \]  
(3.17)  
(3.18)

And therefore the curvature of $X^*$ is obtained as

\[ k_1^* = \frac{1}{L_1 \cos \frac{k_2}{k_1} \phi + L_2 \sin \frac{k_2}{k_1} \phi - \frac{1}{k_1}} \]  
(3.19)

And distance between the opposite points of $(C)$ and $(C^*)$ is

\[ \|X - X^*\| = L_1^2 + L_2^2 = \text{const}. \]  
(3.20)

**References**

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