STRONGLY $\ast$-CLEAN PROPERTIES AND RINGS OF FUNCTIONS

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Abstract. A $\ast$-ring $R$ is called a strongly $\ast$-clean ring if every element of $R$ is the sum of a unit and a projection that commute with each other. In this paper, we explore strong $\ast$-cleanness of rings of continuous functions over spectrum spaces. We prove that a $\ast$-ring $R$ is strongly $\ast$-clean if and only if $R$ is an abelian exchange ring and $C(X)\left(C^*(X)\right)$ is $\ast$-clean, if and only if $R$ is an abelian exchange ring and the classical ring of quotients $q(C(X))$ of $C(X)$ is $\ast$-clean, where $X$ is a spectrum space of $R$.

1. Introduction

Let $R$ be an associative ring with unity. A ring $R$ is called clean if every element of a ring $R$ is the sum of an idempotent and a unit in $R$. If, in addition, these elements are commute, then the ring is called strongly clean. Cleanness of a ring is widely worked since 1977 in many aspects. In 2002, Azarpanah [1], and in 2003, McGovern [11] consider this notion in topological aspects. Let $C(X)$ denote the ring of real valued continuous functions over a topological space $X$. Azarpanah and McGovern independently prove that if $X$ is a completely regular Hausdorff space, then $C(X)$ is clean if and only if $X$ is strongly zero dimensional, if and only if $C^*(X)$ is clean where $C^*(X)$ is the subring of $C(X)$ consisting of all bounded functions in $C(X)$ [1]. On the other hand, in the first section of [12], commutative clean rings are studied by using all maximal ideals and all prime ideals of the ring.

An involution of a ring $R$ is an operation $\ast : R \to R$ such that $(x+y)^\ast = x^\ast + y^\ast$, $(xy)^\ast = y^\ast x^\ast$ and $(x^\ast)^\ast = x$ for all $x, y \in R$. A ring $R$ with involution $\ast$ is called a $\ast$-ring, which has its roots in rings of operators, that is, $\ast$-algebras of operators on a Hilbert space. An element $p$ in a $\ast$-ring $R$ is called a projection if $p^2 = p = p^\ast$. Recently Vas [14] consider cleanness for any $\ast$-ring. A $\ast$-ring $R$ is called $\ast$-clean if each of its elements is the sum of a unit and a projection, and $R$ a strongly $\ast$-clean if each of its elements is the sum of a unit and a projection that commute with each.
other. Also Li and Zhou [8] deal with these notions and answer some questions in [14].

In this paper, we are concerned with the topological properties of strongly $*$-clean rings. Let $Max(R)$ and $Spec(R)$ be the sets of all maximal ideals and all prime ideals of the ring $R$, respectively. Let $J\text{-spec}(R) = \{P \in Spec(R) \mid J(R) \subseteq P\}$. These sets form topological spaces under Zariski topology. We call such topological spaces the spectrum space of $R$. For a $*$-ring $R$, we endow the ring $C(X)$ of continuous functions on $X$ with involution $*$, where $X$ is a spectrum space of $R$. By the help of this, the relationship between strongly $*$-clean rings and the corresponding rings of continuous functions are developed. We then look at the special case of rings of bounded functions. We shall prove that a $*$-ring $R$ is strongly $*$-clean if and only if $R$ is an abelian exchange ring and $C(X)(C^*(X))$ is $*$-clean, where $X$ is a spectrum space of $R$. Along the way, we provide topological characterization of a strongly $*$-clean ring in terms of the classical ring of quotients over its spectrum spaces.

Throughout this paper all rings are associative with unity. We write $J(R)$, $P(R)$ and $U(R)$ for the Jacobson radical, the prime radical and the set of all invertible elements of a ring $R$, respectively. Let $C(X)$ denote the ring of real valued continuous functions over a topological space $X$. Let $S$ and $T$ be two sets. We use $S \cup T$ to denote the set $S \cup T$ with $S \cap T = \emptyset$

2. $*$-Spaces of Prime Ideals

As is well known, $Spec(R)$ is a topological space with Zariski topology. Let $I$ be an ideal of $R$, and let $E_S(I) = \{P \in Spec(R) \mid I \nsubseteq P\}$. Set $V_S(I) = Spec(R) - E_S(I)$, and $V_S(a) := V_S(RaR)$ for any $a \in R$. Then $V_S(I)$ is a closed set of $Spec(R)$. We say that $X$ is a $*$-space provided that $C(X)$ is a $*$-ring.

**Lemma 1.** Let $R$ be a $*$-ring. Then $Spec(R)$ is a $*$-space.

**Proof.** Let $P$ be a prime ideal of $R$. Set $P^* = \{a \in R \mid a^* \in P\}$. It is easy to check that $P^*$ is an ideal of $R$. If $aRb \in P^*$, then $b^*Ra^* \subseteq P$. As $P$ is prime, we see that $b^* \in P$ or $a^* \in P$. Thus, $a \in P^*$ or $b \in P^*$. This implies that $P^*$ is a prime ideal of $R$. Construct a map $*: C(Spec(R)) \to C(Spec(R))$ given by $f \mapsto f^*$, where $f^*(P) = f(P^*)$ for any $P \in Spec(R)$. Clearly, $f^*$ is continuous for any $f \in C(X)$, thus this map is well defined. It is easy to verify that $*$ is a ring morphism. If $f^* = 0$, then for any $P \in Spec(R)$, $f^*(P^*) = 0$, and so $f(P) = 0$. Thus, $f = 0$. That is, $*$ is injective. For any $g \in C(Spec(R))$, we see that $f^* = g$ where $f = g^*$. Therefore $*$ is an involution as $C(Spec(R))$ is commutative. \qed

A ring $R$ is called abelian if every idempotent in $R$ is central. A ring $R$ is an exchange ring provided that for any $a \in R$, there exists an idempotent $e \in aR$ such that $1 - e \in (1-a)R$. For general theory of exchange rings, we refer the reader to [13].
Lemma 2. Let $R$ be a $*$-ring, let $a \in R$, and let $e \in R$ be a projection. If $R$ is an abelian exchange ring, then the following are equivalent:

1. $a - e \in U(R)$, i.e. $a$ is $*$-clean.
2. $V_S(a - 1) \subseteq V_S(e) \subseteq \text{Spec}(R) - V_S(a)$.

Proof. (1) $\Rightarrow$ (2) Set $u := a - e \in U(R)$. Then $1 - a = 1 - e - u$. For any $P \in V_S(a - 1)$, we have $P \not\subseteq V_S(1 - e)$; otherwise, $u = (1 - e) + (a - 1) \in P$. As $R$ is abelian, $\text{Spec}(R) = V_S(e) \cup V_S(1 - e)$, and so $P \not\subseteq V_S(e)$. Thus, $V_S(a - 1) \subseteq V_S(e)$. If $P \in \text{Spec}(R)$ and $P \not\subseteq \text{Spec}(R) - V_S(a)$, then $P \not\subseteq V_S(a)$. This implies that $P \not\subseteq V_S(e)$; otherwise, $u = a - e \in P$. As a result, $V_S(a - 1) \subseteq V_S(e) \subseteq \text{Spec}(R) - V_S(a)$.

(2) $\Rightarrow$ (1) Assume that $V_S(a - 1) \subseteq V_S(e) \subseteq \text{Spec}(R) - V_S(a)$. Let $u = a - e$. Assume that $RuR \neq R$. Then there exists a maximal ideal $M$ of $R$ such that $RuR \subseteq M \subseteq R$. Clearly, $e \in M$ or $1 - e \in M$. Thus, $M \in V_S(e)$ or $M \in V_S(1 - e)$. If $M \in V_S(e)$, then $a = e + u \in M$, whence, $M \in V_S(a)$. This gives a contradiction. If $M \in V_S(1 - e)$, then $a - 1 = (e - 1) + u \in M$, whence, $M \in V_S(a - 1)$. This implies that $M \in V_S(e)$, a contradiction. Thus $RuR = R$. Since $R$ is an exchange ring, analogously to [3, Proposition 17.1.9] that there exists an idempotent $f \in R$ such that $RfR = R$, where $f \in uR$. Since $R$ is abelian, we derive $f = 1$, and so $u \in U(R)$. Therefore $a - e \in R$ is invertible, hence the result holds.

Let $X$ be a topological space. As is well known, a subset $U$ of $X$ is a clopen subset of $X$ if and only if there exists an idempotent $e \in C(X)$ such that $e(x) = 1$ for any $x \in U$ and $e(x) = 0$ for any $x \in X - U$. We say that a subset $U$ of a $*$-space $X$ is $*$-clopen provided that there exists a projection $e \in C(X)$ such that $e(x) = 1$ for any $x \in U$ and $e(x) = 0$ for any $x \in X - U$. A $*$-space $X$ is strongly $*$-zero-dimensional provided that for any disjoint closed subsets $A$ and $B$ there exists a $*$-clopen subset $U$ of $X$ such that $A \subseteq U$ and $B \subseteq X - U$.

Theorem 1. Let $R$ be a $*$-ring. Then $R$ is strongly $*$-clean if and only if

1. $R$ is an abelian exchange ring;
2. $\text{Spec}(R)$ is strongly $*$-zero-dimensional.

Proof. Assume that $R$ is strongly $*$-clean. In view of [8, Lemma 2.1], $R$ is an abelian exchange ring. Let $A$ and $B$ be disjoint closed sets of $\text{Spec}(R)$. Then $A \cap B = \emptyset$. Clearly, there exist two ideals $I$ and $J$ such that $A = V_S(I)$ and $B = V_S(J)$; hence, $V_S(I) \cap V_S(J) = \emptyset$. If $I + J \neq R$, then there exists a maximal ideal $P$ of $R$ such that $I + J \subseteq P \subseteq R$. Hence, $P \subseteq V_S(I + J) = V_S(I) \cap V_S(J)$, a contradiction. This implies that $I + J = R$. Write $a + b = 1$ where $a \in I$ and $b \in J$. By hypothesis, there exists a projection $e \in R$ such that $V_S(a - 1) \subseteq V_S(1 - e) \subseteq \text{Spec}(R) - V_S(a)$.

It is easy to check that

$B = V_S(J) \subseteq V_S(b)$
$= V_S(a - 1) \subseteq V_S(1 - e) \subseteq \text{Spec}(R) - V_S(a)$
$\subseteq \text{Spec}(R) - V_S(I) = \text{Spec}(R) - A$. 


Clearly, $B \subseteq V_S(1 - e)$. As $V_S(1 - e) \subseteq \text{Spec}(R) - A$, we see that $A \subseteq V_S(e)$. Obviously, $\text{Spec}(R) = V_S(e) \uplus V_S(1 - e)$. Define $f : \text{Spec}(R) \to \mathbb{R}$ given by $f(P) = 1$ for any $P \in V_S(e)$ and $f(P) = 0$ for any $P \in V_S(1 - e)$. Then $f \in C(\text{Spec}(R))$.

Clearly, $f^2 = f$. For any $P \in V_S(e)$, we have $e \in P$, and so $e \in P^*$. This implies that $P^* \subseteq V_S(e)$. Thus, $f^*(P) = f(P^*) = f(P) = 1$. Likewise, $f^*(P) = f(P) = 0$ for any $P \in V_S(1 - e)$. Therefore $f = f^*$. This shows that $V_S(e)$ is a $*$-clopen set. Therefore $\text{Spec}(R)$ is strongly $*$-zero-dimensional.

Conversely assume that (1) and (2) hold. For any $a \in R$, we see that $V_S(a) \cap V_S(1 - a) = \emptyset$, and so there exists a $*$-clopen $U$ such that $V_S(a - 1) \subseteq U \subseteq \text{Spec}(R) - V_S(a)$. Thus, we have a projection $f \in C(\text{Spec}(R))$ such that $f(P) = 1$ for any $P \in U$ and $f(P) = 0$ for any $P \in \text{Spec}(R) - U$. As $U$ is clopen and the prime radical $P(R)$ is nil, analogously to [3, Lemma 17.1.10], we can find an idempotent $e \in R$ such that $U = V_S(e)$.

Now we claim that $e$ is a projection. For any $P \in V_S(e)$, we see that $f(P) = 1$, and so $f^*(P) = f(P^*) = f(P) = 1$. This implies that $P^* \subseteq U = V_S(e)$, and so $e \in P^*$. Hence, $e^* \in P$, and then $P \subseteq V_S(e^*)$. As a result, $V_S(e) \subseteq V_S(e^*)$. For any $P \in V_S(1 - e)$, we see that $f(P) = 0$, and so $f(P^*) = f^*(P) = f(P) = 0$, and so $P^* \subseteq V_S(1 - e)$. This implies that $1 - e \in P^*$, and so $1 - e^* \in P$. Hence, $P \subseteq V_S(1 - e^*)$. This shows that $V_S(1 - e) \subseteq V_S(1 - e^*)$. As $\text{Spec}(R) = V_S(e) \uplus V_S(1 - e) = V_S(e^*) \uplus V_S(1 - e^*)$, we get $V_S(e) = V_S(e^*)$ and $V_S(1 - e) = V_S(1 - e^*)$. For any $P \in \text{Spec}(R)$, if $P \in V_S(e)$, then $P \subseteq V_S(e^*)$, and so $e, e^* \in P$. Thus, $e - e^* \in P$. If $P \in V_S(1 - e)$, then $P \subseteq V_S(1 - e^*)$, and so $1 - e, 1 - e^* \in P$. This implies that $e - e^* = (1 - e^*) - (1 - e) \in P$. Therefore $e - e^* \in P(R)$. As $P(R)$ is nil, we see that $(e - e^*)^n = 0$ for some $n \in \mathbb{N}$. As $e - e^* = (e - e^*)^3$, we see that $e = e^*$. That is, $e \in R$ is a projection. In view of Lemma 2, we complete the proof.

Recall that two subsets $A$ and $B$ of $X$ is said to be completely separated if there exists $f \in C(X)$ such that $0 \leq f \leq 1$, $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. Let $X$ be a topological space, and let $A$ be a subset of $X$. Then $A$ is a zero set in $X$ provided that there exists an element $f \in C(X)$ such that $A = \{ x \in X \mid f(x) = 0, \}$, and denote $A$ by $Z(f)$. Every zero set is a closed set, but the converse does not always hold.

**Lemma 3.** Let $X$ be a $*$-space. Then $X$ is strongly $*$-zero-dimensional if and only if

1. $C(X)$ is $*$-clean;
2. Any two disjoint closed sets of $X$ are completely separated.

**Proof.** Suppose that $X$ is strongly $*$-zero-dimensional. Then any disjoint closed sets of $X$ are completely separated. Let $f \in C(X)$. Let $A = f^{-1}(0)$ and $B = f^{-1}(1)$. Since every zero set of $X$ is closed, we see that $A$ and $B$ are both disjoint closed sets of $X$. By hypothesis, there exists a $*$-clopen set $U$ of $X$ such that $A \subseteq U$ and $B \subseteq X - U$. Let $e \in C(X)$ be a projection such that $e(x) = 1$ for any $x \in U$ and $e(x) = 0$ for any $x \in X - U$. Let $u = f - e$. For any $x \in U$, $e(x) = 1$. If $f(x) = 1$, then
then $x \in B$, and so $x \in X - U$, a contradiction. Thus, $f(x) \neq 1$. This implies that $u(x) \neq 0$ for any $x \in U$. If $x \in X - U$, then $e(x) = 0$. If $f(x) = 0$, then $x \in A \subseteq U$, a contradiction, and so $f(x) \neq 0$. This implies that $u(x) \neq 0$ for any $x \in X - U$. Therefore $u(x) \neq 0$ for any $x \in X$. Hence $u^{-1}(x) := \frac{1}{u(x)}$ for any $x \in X$. That is, $u \in C(X)$ is invertible. Therefore $f = e + u \in C(X)$ is $*$-clean.

Conversely, assume that (1) and (2) hold. Let $A$ and $B$ be disjoint closed sets. Then $A$ and $B$ are completely separated. In light of [5, Theorem 1.15], $A$ and $B$ are contained in disjoint zero sets. Thus, we can find some $f_1, f_2 \in C(X)$ such that $A \subseteq Z(f_1), B \subseteq Z(f_2)$ and $Z(f_1) \cap Z(f_2) = \emptyset$. This shows that $|f_1| + |f_2| > 0$. Choose $h = \frac{|f_1|}{|f_1| + |f_2|} \in C(X)$. Since $C(X)$ is $*$-clean, there exist a projection $e \in C(X)$ and a unit $u \in C(X)$ such that $h = e + u$. For any $x \in X$, $e(x) \cdot e(x) = e(x)$, and so $e(x) = 0$ or $e(x) = 1$. Set $U = \{x \in X \mid e(x) = 0\}$ and $V = \{x \in X \mid e(x) = 1\}$. Then $X = U \bigcup V$. As $U$ and $V$ are closed, and so $V$ is clopen. Further, $V$ is $*$-clopen. As $u \in C(X)$ is a unit, we see that $u(x) \neq 0$ for all $x \in X$. For any $x \in A$, we see that $f_1(x) = 0$, and so $h(x) = 0$. Thus, $e(x) \neq 0$ as $u(x) \neq 0$, and then $x \in V$. That is, $A \subseteq V$. For any $x \in B$, $f_2(x) = 0$, and so $h(x) = 1$. This implies that $e(x) = 0$; hence, $x \in X - V$. Thus, $B \subseteq X - V$. Therefore $X$ is strongly $*$-zero-dimensional.

**Theorem 2.** Let $R$ be a $*$-ring. Then $R$ is strongly $*$-clean if and only if

1. $R$ is an abelian exchange ring;
2. $C(Spec(R))$ is $*$-clean.

**Proof.** If $R$ is strongly $*$-clean, then (1) and (2) follows from Theorem 1 and Lemma 3.

Conversely, assume that (1) and (2) hold. Then $R$ is strongly clean. In view of [3, Lemma 17.1.12], $Spec(R)$ is strongly zero dimensional. Thus, for any disjoint closed sets $A$ and $B$ of $Spec(R)$, there exists a clopen $U$ such that $A \subseteq U$ and $B \subseteq Spec(R) - U$. It follows from Urysohn’s Lemma, there exists a continuous function $f: Spec(R) \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. Thus, $A$ and $B$ are completely separated. By virtue of Lemma 3, $Spec(R)$ is strongly $*$-zero-dimensional. Therefore we complete the proof from Theorem 1.

The condition “$C(Spec(R))$ is $*$-clean" in Theorem 2 is necessary, as the following shows.

**Example 1.** Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then the map $*: R \to R, (a, b)^* = (b, a)$ is an involution. Obviously, $R$ is an abelian exchange ring. Further, $R$ is a commutative $*$-ring. But $R$ is not strongly $*$-clean, as the idempotent $e = (1, 0) \in R$ is not a projection (see [8, Theorem 2.2]).

3. Extensions to $*$-Subspaces

Let $I$ be an ideal of a $*$-ring $R$, and let $E_M(I) = \{P \in Max(R) \mid I \nsubseteq P\}$. Set $V_M(I) = Max(R) - E_M(I)$. Then $Max(R)$ is a topological space with closed
sets \( V_M(I) \). Denote \( M^* = \{ a \in R \mid a^* \in M \} \) for a maximal ideal \( M \). Clearly, \( M \in \text{Max}(R) \) if and only if \( M^* \in \text{Max}(R) \). Construct a map \( * : C(\text{Max}(R)) \rightarrow C(\text{Max}(R)) \) given by \( f \mapsto f^* \), where \( f^*(M) = f(M^*) \) for any \( M \in \text{Max}(R) \). As in the proof of Lemma 1, \( * \) is an anti-automorphism of \( C(\text{Max}(R)) \). Therefore \( \text{Max}(R) \) is a \( * \)-space.

**Lemma 4.** Let \( R \) be a \( * \)-ring. Then \( R \) is strongly \( * \)-clean if and only if

1. \( R \) is an abelian exchange ring;
2. \( \text{Max}(R) \) is strongly \( * \)-zero-dimensional.

**Proof.** Suppose that \( R \) is strongly \( * \)-clean. Then it is an abelian exchange ring. As in the proof of Lemma 1, \( a - e \in U(R) \) and only if \( V_M(a - 1) \subseteq \text{Max}(R) - V_M(e) \), where \( e \in R \) is a projection. Let \( A \) and \( B \) be disjoint closed sets of \( \text{Max}(R) \). Analogously to the discussion in Theorem 1, there exists a projection \( e \in R \) such that \( A \subseteq V_M(e) \) and \( B \subseteq V_M(1 - e) \). Define \( f : \text{Max}(R) \rightarrow \mathbb{R} \) given by \( f(M) = 1 \) for any \( M \in V_M(e) \) and \( f(M) = 0 \) for any \( M \in V_M(1 - e) \). Then \( f \in C(\text{Max}(R)) \). Similar to the consideration in Theorem 1, \( V_S(e) \) is a \( * \)-clopen set. Therefore \( \text{Max}(R) \) is strongly \( * \)-zero-dimensional.

Conversely, assume that (1) and (2) hold. Then \( R \) is clean. In view of [3, Theorem 17.1.13], \( R \) is a pm ring, where a ring is a pm ring provided that each prime ideal is contained in exactly one maximal ideal. Thus, there exists a map \( \varphi : \text{Spec}(R) \rightarrow \text{Max}(R) \), \( \varphi(P) = M \), where \( M \) is the unique maximal ideal such that \( P \subseteq M \). It is easy to check that \( \varphi(V_M(I)) = V_M(I) \). This shows that \( \varphi \) is continuous. For any disjoint closed sets \( A, B \subseteq \text{Spec}(R) \), there exist two ideals \( I \) and \( J \) of \( R \) such that \( A = V_S(I) \) and \( B = V_S(J) \). Hence, \( \varphi(A) \) and \( \varphi(B) \) are both closed. As \( V_S(I) \cap V_S(J) = \emptyset \), we see that \( V_S(I + J) = \emptyset \); hence, \( I + J = R \). Thus, we infer that \( V_M(I) \cap V_M(J) = V_M(I + J) = V_M(R) = \emptyset \). This shows that \( \varphi(A) \) and \( \varphi(B) \) are disjoint closed sets of \( \text{Max}(R) \). By hypothesis, \( \text{Max}(R) \) is strongly \( * \)-zero-dimensional, there exist disjoint \( * \)-clopen sets \( U, V \subseteq \text{Max}(R) \) such that \( V_M(I) \subseteq U \), \( V_M(J) \subseteq V \). Clearly, \( A \subseteq \varphi^-(\varphi(A)) \subseteq \varphi^-(U) \) and \( B \subseteq \varphi^-(\varphi(B)) \subseteq \varphi^-(V) \). Clearly, \( \varphi^-(U) \) and \( \varphi^-(V) \) are clopen. For any \( P \in \varphi^-(U) \cap \varphi^-(V) \), there exists a unique \( M \in \text{Max}(R) \) such that \( P \subseteq M \). Hence, \( M \in U \cap V \), a contradiction. This shows that \( \varphi^-(U) \cap \varphi^-(V) = \emptyset \).

As \( U \) is a \( * \)-clopen set of \( \text{Max}(R) \), there exists a projection \( e \in C(\text{Max}(R)) \) such that \( e(x) = 1 \) for any \( x \in U \) and \( e(x) = 0 \) for any \( x \in \text{Max}(R) - U \). Construct a function \( f : \text{Spec}(R) \rightarrow \mathbb{R} \) given by \( P \mapsto e\varphi(P) \) for any \( P \in \text{Spec}(R) \). Then \( f \in C(\text{Spec}(R)) \) is a projection. Further, we see that \( f(y) = e\varphi(y) = 1 \) for any \( y \in \varphi^-(U) \) and \( f(y) = e\varphi(y) = 0 \) for any \( y \in \text{Spec}(R) - \varphi^-(U) \). This implies that \( \varphi^-(U) \) is \( * \)-clopen. Likewise, \( \varphi^-(V) \) is \( * \)-clopen. Therefore \( \text{Spec}(R) \) is strongly \( * \)-zero-dimensional, and thus completing the proof by Theorem 1.

**Theorem 3.** Let \( R \) be a \( * \)-ring. Then \( R \) is strongly \( * \)-clean if and only if

1. \( R \) is an abelian exchange ring;
2. \( C(\text{Max}(R)) \) is \( * \)-clean.
Example 2. Let 

\[ e = 17.1.13 \]

Max add as the identity. Then

According to Lemma 3, \( Max(R) \) is strongly \(*\)-zero-dimensional. This completes the proof by Lemma 4.

The following observation is crucial.

**Example 2.** Let \( R = \left\{ \frac{m}{n} \in \mathbb{Q} \mid m, n \in \mathbb{Z}, (n, 6) = 1 \right\} \). We choose the involution as the identity. Then \( R \) is a commutative ring. Clearly, \( Max(R) = \{2R, 3R\} \). As \( Max(R) \) is a finite set, it follows from [5, Remark 2.3] that \( C(Max(R)) \) is \(*\)-clean. But \( R \) is not strongly \(*\)-clean. In fact, \( R \) is not an exchange ring.

Clearly, the Jacobson radical \( J(R) \) is semiprime, and so \( J(R) \) is the intersection of some prime ideals. Thus, \( J(R) = \bigcap_{P \in J-spec(R)} P \). Let \( I \) be an ideal of \( R \), and let \( F(I) = \{ P \in J-spec(R) \mid I \not\subseteq P \} \). Then \( F(R) = J-spec(R), F(0) = \emptyset, F(I) \cap F(J) = F(IJ) \) and \( \bigcup I F(I) = F(\bigcup I I) \). So \( J-spec(R) \) is a topological subspace of \( Spec(R) \), where \( \{ F(I) \mid I \subseteq R \} \) is the collection of its open sets. Let \( W(I) = J-spec(R) - F(I) \). Then \( W(I) = \{ P \in J-spec(R) \mid I \subseteq P \} \) is the collection of its closed sets. Let \( R \) be a \(*\)-ring. As in the proof of Lemma 1, \( J-spec(R) \) is a \(*\)-space. The next aim is to investigate strong \(*\)-cleanness of \(*\)-rings by such \(*\)-subspaces. The following observation will clear our path.

**Lemma 5.** Let \( R \) be a \(*\)-ring. Then \( R \) is strongly \(*\)-clean if and only if

(1) \( R \) is an abelian exchange ring;

(2) \( R/J(R) \) is strongly \(*\)-clean.

**Proof.** One direction is obvious. Conversely, assume that (1) and (2) hold. For any \( a \in R \), there exists a projection \( \overline{f} = f + J(R) \in R/J(R) \) and a unit \( \overline{e} \in R/J(R) \) such that \( \overline{a} = \overline{e} + \overline{a} \). As \( f - f = 0 \) in \( J(R) \), by hypothesis, there exists an idempotent \( e \in R \) such that \( f - e \in J(R) \). Since every unit lifts modulo \( J(R) \), we may assume that \( u \in U(R) \). Thus, \( a = e + u + r \) for some \( r \in J(R) \). Set \( v = u + r \). Then \( a = e + v \) with \( e = e^2 \in R, v \in U(R) \). As \( R \) is abelian, \( ae = ea \) and \( ae^* = e^*a \). Further, \( e - e^* = f - f \in J(R) \).

Let \( p = 1 + (e^* - e)(e^* - e) \). As \( ae = ea, ae^* = e^*a \), we see that \( ap = pa \). Clearly, \( p \in U(R) \). Write \( q = p^{-1} \). Then \( p^* = p \), and so \( q^* = q \). Further, \( ep = e(1 - e - e^* + ee^* + e^* e) = ee^* e = (1 - e - e^* + ee^* + e^* e) e = pe \). Thus, we see that \( eq = q \) and \( e^* q = q e^* \). Set \( g = e^* q \). Then \( g^2 = e^* q e^* q = q e^* e^* q = q e^* e^* q = e^* q = g \). In addition, \( g^* = q^* e^* = ee^* q = g \), i.e., \( g \in R \) is a projection. As \( aq = qa \), we see that \( ag = ga \). One easy check that \( eg = g \) and \( ge = ee^* q e = ee^* q = epq = e \). This
implies that \( e - g = e - ee^*q = e(ep - ee^*)q = ee^*(e - e^*)q \in J(R) \).
Therefore \( a = e + v = g + (e - g) + v \). Clearly, \( (e - g) + v \in U(R) \). Let \( w = (e - g) + v \).
Then \( a = g + w, \ g^2 = g = g^* \), \( w \in U(R) \) and \( ag = ga \). Therefore \( R \) is strongly *-clean.

\[ \square \]

**Theorem 4.** Let \( R \) be a *-ring. Then \( R \) is strongly *-clean if and only if

1. \( R \) is an abelian exchange ring;
2. \( C(J \text{-spec}(R)) \) is *-clean.

**Proof.** Construct a map \( \varphi : J \text{-spec}(R) \to \text{Spec}(R/J(R)) \) given by \( P \mapsto \overline{P} \) for any \( P \in J \text{-spec}(R) \). Then \( \varphi \) is a continuous map. If \( \varphi(P) = \varphi(Q) \), then \( \overline{P} = \overline{Q} \). For any \( p \in P \), write \( p + J(R) = q + J(R) \) for some \( q \in Q \). This implies that \( p \in q + J(R) \subseteq Q \) and so \( P \subseteq Q \). Likewise, \( Q \subseteq P \). Hence, \( P = Q \), and so \( \varphi \) is injective. For any \( \overline{P} \in \text{Spec}(R/J(R)) \), then \( P \in J \text{-spec}(R) \), and then \( \varphi \) is surjective. That is, \( \varphi \) is bijective. Further, one can easily check that \( \varphi \) is a homeomorphism. Construct a map \( \phi : C(J \text{-spec}(R)) \to C(\text{Spec}(R/J(R))) \) given by \( f \mapsto f \varphi^{-1} \) for any \( f \in C(J \text{-spec}(R)) \). In addition, \( \varphi(f^*) = (\varphi(f))^* \). Therefore \( C(J \text{-spec}(R)) \) and \( C(\text{Spec}(R/J(R))) \) are *-isomorphic.

If \( R \) is strongly *-clean, then \( R \) is an abelian exchange ring. In view of Lemma 5, \( R/J(R) \) is strongly *-clean. It follows from Theorem 2, \( C(\text{Spec}(R/J(R))) \) is strongly *-clean, and therefore so is \( C(J \text{-spec}(R)) \). Conversely, assume that (1) and (2) hold. Then \( R/J(R) \) is an abelian exchange ring and \( C(\text{Spec}(R/J(R))) \) is strongly *-clean. In light of Theorem 2, \( R/J(R) \) is strongly *-clean. Therefore \( R \) is strongly *-clean by Lemma 5.

\[ \square \]

**Corollary 1.** Let \( R \) be a *-ring. Then \( R \) is strongly *-clean if and only if

1. \( R \) is an abelian exchange ring;
2. \( J \text{-spec}(R) \) is strongly *-zero-dimensional.

**Proof.** Suppose that \( R \) is strongly *-clean. Then \( R \) is an abelian exchange ring. It follows by Theorem 4 that \( C(J \text{-spec}(R)) \) is *-clean. Analogously to the proof of Theorem 2, any two disjoint closed sets of \( J \text{-spec}(R) \) are completely separated. Therefore \( J \text{-spec}(R) \) is strongly *-zero-dimensional from Lemma 3.

Conversely, assume that (1) and (2) hold. In view of Lemma 3, \( C(J \text{-spec}(R)) \) is *-clean. Hence the result follows by Theorem 4.

\[ \square \]

Combining Theorems 2, 3 and 4, we come now to the following main result.

**Theorem 5.** Let \( R \) be a *-ring, and let \( X \) be a spectrum space of \( R \). Then \( R \) is strongly *-clean if and only if

1. \( R \) is an abelian exchange ring;
2. \( C(X) \) is *-clean.
4. THE RING OF BOUNDED CONTINUOUS FUNCTIONS

Let $X$ be a topological space. $C^*(X)$ denote the subring of $C(X)$ of all bounded functions. In the following lemma we follow the technique of [1, Lemma 2.1].

Lemma 6. Let $X$ be a $*$-space. Then $f \in C(X)$ is $*$-clean if and only if there exists a $*$-clopen set $U$ in $X$ such that $f^{-1}(1) \subseteq U \subseteq X - Z(f)$.

Proof. Let $f \in C(X)$ be $*$-clean. Then there exists a projection $e \in C(X)$ such that $f - e \in U(C(X))$. Set $U = Z(e)$. Clearly, $X = Z(e) \cup Z(1 - e)$, $e(Z(e)) = \{0\}$ and $e(Z(1 - e)) = \{1\}$. Thus, $U$ is a $*$-clopen set. One easily checks that $f^{-1}(1) \subseteq U \subseteq X - Z(f)$. Conversely, assume that $f^{-1}(1) \subseteq U \subseteq X - Z(f)$ for a $*$-clopen set $U$. Then $U = Z(e)$ for some projection $e$. Construct $u : X \to \mathbb{R}$ given by $u(x) = f(x)$ for any $x \in Z(e)$ and $u(x) = f(x) - 1$ for any $x \in X - Z(e)$. Then $u = f - e$. If $x \in Z(e)$, then $x \not\in Z(f)$, and so $f(x) \neq 0$. Hence, $u(x) \neq 0$. If $x \in X - Z(e)$, then $x \not\in f^{-1}(1)$, and so $f(x) \neq 1$. This implies that $u(x) \neq 0$. Consequently, $u \in U(C(X))$, as required.

Lemma 7. Let $R$ be a $*$-ring, and let $X$ be a spectrum space of $R$. Then $C(X)$ is $*$-clean if and only if so is $C^*(X)$.

Proof. For any $f \in C^*(X)$, we define $f^* : X \to \mathbb{R}$ given by $f^*(P) = f(P^*)$ for any $P \in X$. One easily checks that $f^* \in C^*(X)$. This induces an involution $*: C^*(X) \to C^*(X)$ given by $f \mapsto f^*$. Therefore $C^*(X)$ is a $*$-ring.

Suppose that $C(X)$ is $*$-clean. Let $f \in C^*(X)$. Choose $A = \{x \in X | f(x) \geq \frac{2}{3}\}$ and $B = \{x \in X | f(x) \leq \frac{1}{3}\}$. Construct a function $g \in C(X)$ such that $g(x) = 1, x \in A; g(x) = 0, x \in B$ and $g(x) = \frac{1}{2}$, otherwise. Then $g \in C(X)$ is $*$-clean. In view of lemma 6, there exists a $*$-clopen set $U$ in $X$ such that $g^{-1}(1) \subseteq U \subseteq X - Z(g)$. Write $U = Z(e)$ for a projection $e \in C(X)$. Construct $u : X \to \mathbb{R}$ given by $u(x) = f(x)$ for any $x \in Z(e)$ and $u(x) = f(x) - 1$ for any $x \in X - Z(e)$. Then $u = f - e$. If $x \in Z(e)$, then $x \not\in Z(g)$, and so $g(x) \neq 0$. Thus, $x \not\in B$, and so $f(x) \neq 0$. This shows that $u(x) \neq 0$. If $x \in X - Z(e)$, then $x \not\in g^{-1}(1)$, and so $g(x) \neq 1$. Hence, $x \not\in A$. This shows that $f(x) \neq 1$. This implies that $u(x) \neq 0$. In addition, $u \in C^*(X)$. Therefore $u \in U(C^*(X))$, and thus $f \in C^*(X)$ is $*$-clean, as desired.

We now assume $C^*(X)$ is $*$-clean. Let $f \in C(X)$. Set $h(x) = \{ -1, \quad \text{if } f(x) < -1; 
1, \quad \text{if } f(x) \geq -1. \}$

Choose $g(x) = \{ h(x), \quad \text{if } h(x) < 1; 
1, \quad \text{if } h(x) \geq 1. \}$

Then $g \in C^*(X)$. By hypothesis, $g$ is $*$-clean. This implies that $g \in C(X)$ is $*$-clean. In view of Lemma 6, there exists a $*$-clopen set $U$ in $X$ such that $g^{-1}(1) \subseteq U \subseteq X - Z(g)$. It is easy to check that $f^{-1}(1) \subseteq g^{-1}(1)$ and $X - Z(g) \subseteq X - Z(f)$. Therefore $f^{-1}(1) \subseteq U \subseteq X - Z(f)$. This completes the proof by Lemma 6. \[\square\]
Theorem 6. Let $R$ be a $*$-ring, and let $X$ be a spectrum space of $R$. Then $R$ is strongly $*$-clean if and only if

1. $R$ is an abelian exchange ring;
2. $C^*(X)$ is $*$-clean.

Proof. In view of Lemma 7, $C(X)$ is strongly $*$-clean if and only if so is $C^*(X)$. Therefore we complete the proof by Theorem 5.

The Stone-Cech compactification $\beta X$ of a topological space $X$ is the largest compact Hausdorff space "generated" by $X$, in the sense that any map from $X$ to a compact Hausdorff space factors through $\beta X$ (in a unique way). That is, $\beta X$ is a compact Hausdorff space together with a continuous map from $X$ and has the following universal property: any continuous map $f : X \to K$, where $K$ is a compact Hausdorff space, lifts uniquely to a continuous map $\beta f : \beta X \to K$.

Corollary 2. Let $R$ be a $*$-ring, and let $X$ be a spectrum space of $R$. Then $R$ is strongly $*$-clean if and only if

1. $R$ is an abelian exchange ring;
2. The Stone-Cech compactification $\beta X$ of $X$ is strongly $*$-zero dimensional.

Proof. Suppose that $R$ is strongly $*$-clean. Then $R$ is an abelian exchange ring. In view of [5, Remark 6.6], $C(\beta X) \cong C^*(X)$. Thus, $C(\beta X)$ is $*$-clean by Theorem 6. Hence, $\beta X$ is a $*$-space. Clearly, $C(\beta X)$ is a commutative clean ring. According to [1, Theorem 2.5], $\beta X$ is strongly zero dimensional. This shows that any two disjoint closed sets of $\beta X$ are completely separated. Therefore $\beta X$ of $X$ is strongly $*$-zero dimensional by Lemma 3.

Conversely, assume that (1) and (2) hold. In light of Lemma 3, $C(\beta X)$ is $*$-clean. By virtue of [5, Remark 6.6], $C^*(X)$ is $*$-clean. Accordingly, $R$ is strongly $*$-clean from Theorem 6.

Corollary 3. Let $R$ be a $*$-ring, and let $X$ be a spectrum space of $R$. Then $R$ is strongly $*$-clean if and only if

1. $R$ is an abelian exchange ring;
2. $\text{Max}(C^*(X))$ is strongly $*$-zero dimensional.

Proof. By virtue of [5, 14.8] or [10, p. 463], the prime ideals containing a given ideal forms a chain in $C^*(X)$, and so $C^*(X)$ is a pm-ring. In view of [3, Corollary 17.1.14], $C^*(X)$ is $*$-clean. This completes the proof by Theorem 6.

5. Strong $*$-Cleaness of $q(X)$

Let $R$ be a commutative $*$-ring with an identity, and let $q(R)$ be the classical ring of quotients of $R$. We say that $x \in R$ is self-adjoint provided that $x^* = x$. Construct a ring morphism $*: q(R) \to q(R), \frac{r}{s} \mapsto \frac{r^*}{s^*}$. Then $*$ is also an involution of $q(R)$. Thus, $q(R)$ is a $*$-ring.
Let $N_D(R)$ denote the set of all nonzero divisors of $R$, and let $N_D(X) := N_D(C(X))$ for a topological space $X$.

**Lemma 8.** Let $R$ be a commutative $*$-ring. If $e \in q(R)$ is self-adjoint, then there exist self-adjoint $a, b \in R$ such that $e = \frac{a}{b}$.

**Proof.** Write $e = \frac{a}{b}$. As $e \in q(R)$ is self-adjoint, $e^* = \left(\frac{a}{b}\right)^* = \frac{a^*}{b^*} = \frac{a}{b}$. Thus, $c^*d = d^*c$. Clearly, $d, d^* \in N_D(R)$; hence, $e = \frac{cd^*}{d^*c}$. Set $a = cd^*$ and $b = dd^*$. Then $a^* = (cd^*)^* = a$ and $b^* = b$. That is, $a, b \in R$ are self-adjoint. In addition, $e = \frac{a}{b}$, as required. □

**Lemma 9.** Let $R$ be a commutative $*$-ring. Then the following are equivalent:

1. $q(R)$ is $*$-clean.
2. For any $a, b \in R$ with $\alpha + \beta \in N_D(R)$, there exist self-adjoint $x \in aR, y \in bR$ such that $x + y \in N_D(R)$ and $xy = 0$.
3. For any $a, b \in R$ with $\alpha + \beta \in N_D(R)$, there exist $x \in aR, y \in bR$ such that $x + y \in N_D(R), xy = 0$ and $x^*y$ is self-adjoint.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $a + b \in N_D(R)$ with $a, b \in R$. Then there exists some $\alpha \in q(R)$ such that $\alpha a + \beta a = 1$. Since $q(R)$ is $*$-clean, we can find a projection $e \in q(R)$ such that $e \in baq(R) \subseteq bq(R)$ and $1 - e \in aaq(R) \subseteq aq(R)$. Write $e = \frac{ba}{T} = \frac{baq}{Tq}$. Set $w = bst^*$ and $u = tt^*$. Then $e = \frac{uw}{w}$, where $w, u \in R$ are self-adjoint and $w \in bR$. Analogously, $1 - e = \frac{zt}{t}$, where $z, t \in R$ are self-adjoint and $z \in aR$. Obviously, $\frac{w}{w} + \frac{zt}{t} = 1$, and so $w(zt) = ut$. Choose $x = wt$ and $y = zu$. Then $x + y = ut \in N_D(R)$. Further, $xy = (wz)(ut) = 0$ and $x, y \in R$ are self-adjoint.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Suppose that $\frac{a}{b} + \frac{c}{d} = 1$ in $q(R)$. Then $a + b = s \in N_D(R)$. By hypothesis, there exist $x \in aR, y \in bR$ such that $x + y \in N_D(R), xy = 0$ and $x^*y$ is self-adjoint. Let $e = \frac{x^*y}{x+y}$. Then $e(1 - e) = \frac{xx^*y}{x+y} = 0$, and so $e = e^2 \in q(R)$ is an idempotent. Since $x^*y \in R$ is self-adjoint, we see that $(x^*y)^* = x^*y = xy^*$, and so $e^* = \frac{x^*y}{x^*y} = \frac{x^*y}{x+y} = e$; hence, $e \in q(R)$ is a projection. Moreover, $e = \frac{x^*y}{x+y} \in (\frac{a}{b})q(R)$ and $1 - e = \frac{y}{x+y} \in (\frac{c}{d})q(R)$. Therefore $q(R)$ is strongly $*$-clean. □

Let $X$ be a $*$-space. Then $C(X)$ is a $*$-ring. We denote $q(C(X))$ by $q(X)$, and so $q(X)$ is a $*$-ring. We say that $U$ is a $*$-zero set of $X$ provided that there exists a self-adjoint $f \in C(X)$ such that $A = Z(f)$. Let $A$ be a subset of $X$. We say that $A$ is nowhere dense if every open set of $X$ contains an open subset that is disjoint from $A$. This is equivalent to saying that the closure of $A$ contains no open set of $A$ which is not empty. Clearly, every subset of a nowhere dense set is nowhere dense. We say that $A$ is dense in $X$ if $X - A$ is nowhere dense.

Recall that a topological space $X$ is completely regular if for every point and a closed set not containing the point, there is a continuous function that has value 0 at the given point and value 1 at each point in the closed set. Almost every topological
By hypothesis, there exists an open subset.

Lemma 10. Let X be a completely regular space, and let \( f \in C(X) \). Then the following are equivalent:

1. \( f \in N_D(X) \).
2. \( Z(f) \) is nowhere dense.

Proof. (2) \( \Rightarrow \) (1) Assume that \( f \phi = 0 \) for a \( \phi \in C(X) \). Assume that \( Z(\phi) \neq X \).

Assume that there exists an open subset \( B \) of \( X - Z(\phi) \) such that \( Z(f) \cap B = \emptyset \). Thus, we can find \( x \in B \) such that \( x \notin Z(f) \). This implies that \( f(x), \phi(x) \neq 0 \). This yields that \( f\phi \neq 0 \), a contradiction. Thus, \( Z(\phi) = X \), and so \( \phi = 0 \). This means that \( f \in N_D(X) \).

(1) \( \Rightarrow \) (2) Let \( C \) be an open set of \( X \), and let \( B = C \cap (X - Z(f)) \). If \( B \neq \emptyset \), then \( B \) is an open subset of \( C \).

In addition, \( Z(f) \cap B = \emptyset \). If \( B = \emptyset \), then we have \( C \subseteq Z(f) \), and so \( f(C) = 0 \). Choose \( a \in C \).

Since \( X \) is a completely regular space, we can find some \( g \in C(X) \) such that \( g(x) = 0 \) for any \( x \in X - C \) and \( g(a) = 1 \).

This implies that \( fg = 0 \). By hypothesis, \( g = 0 \), a contradiction. Therefore we complete the proof.

Theorem 7. Let \( X \) be a completely regular \( * \)-space. Then the following are equivalent:

1. \( g(X) \) is \( * \)-clean.
2. For any zero sets \( A \) and \( B \) of \( X \) such that \( A \cap B \) is nowhere dense, there exist \( * \)-zero sets \( U, V \) such that \( A \subseteq U, B \subseteq V \) such that \( U \cap V \) is nowhere dense and \( U \cup V = X \).

Proof. (1) \( \Rightarrow \) (2) For any zero sets \( A \) and \( B \) of \( X \) such that \( A \cap B \) is nowhere dense, we can write \( A = Z(f) \) and \( B = Z(g) \).

Since \( Z(f^2 + g^2) = Z(f) \cap Z(g) = U \cap V \) is nowhere dense, it follows from Lemma 10 that \( f^2 + g^2 \in N_D(X) \). In view of Lemma 9, there exist self-adjoint \( h \in f^2C(X), k \in g^2C(X) \) such that \( h + k \in N_D(X) \) and \( hk = 0 \).

Let \( U = Z(h) \) and \( V = Z(k) \). Then \( A \subseteq U, B \subseteq V \). In addition, \( U \cup V = Z(h) \cup Z(k) = Z(hk) = Z(0) = X \).

Further, \( U \cap V = Z(h) \cap Z(k) = Z(h^2 + k^2) \). As \( h^2 + k^2 = (h + k)^2 \), we see that \( U \cap V = Z((h + k)^2) = Z(h + k) \) is nowhere dense from Lemma 10.

Since \( h, k \in q(X) \) are self-adjoint, \( U \) and \( V \) are both \( * \)-zero sets, as required.

(2) \( \Rightarrow \) (1) Let \( f, g \in C(X) \) such that \( f + g \in N_D(X) \). Let \( A = Z(f) \) and \( B = Z(g) \).

Then \( A \cap B = Z(f) \cap Z(g) \subseteq Z(f + g) \); hence, \( A \cap B \) is nowhere dense from Lemma 10. By hypothesis, there exist \( * \)-zero sets \( U, V \) such that \( A \subseteq U, B \subseteq V \) such that \( U \cap V \) is nowhere dense and \( U \cup V = X \).

Thus, we can find self-adjoint \( h, k \in C(X) \) such that \( U = Z(h) \) and \( V = Z(k) \).

Set \( \varphi = fh \in fC(X) \) and \( \psi = gk \in gC(X) \). Then \( Z(\varphi) = Z(fh) = Z(f) \cup Z(h) = Z(h) \).

Likewise, \( Z(\psi) = Z(k) \).

Thus, \( Z(\varphi^2 + \psi^2) = Z(\varphi) \cap Z(\psi) = Z(h) \cap Z(k) \) is nowhere dense, and so...
\[ \varphi^2 + \psi^2 \in N_D(X) \text{ from Lemma 10. As } Z(\varphi^2\psi^2) = Z(\varphi) \cup Z(\psi) = Z(h) \cup Z(k) = X, \text{ we see that } \varphi^2\psi^2 = 0. \] In addition, it follows from \( Z(hk) = Z(h) \cup Z(k) = X \) that \( hk = 0 \). Therefore \( (\varphi^2)^*\psi^2 = (fg)^2hk(hk) = 0 \). According to Lemma 9, \( q(X) \) is *-clean.

**Lemma 11.** Let \( X \) be a completely regular *-space. Then \( C(X) \) is *-clean if and only if

1. \( X \) is strongly zero-dimensional;
2. \( q(X) \) is *-clean.

**Proof.** Suppose that \( C(X) \) is *-clean. Then \( q(X) \) is *-clean. By [1, Theorem 2.5], \( X \) is strongly zero-dimensional, as desired.

Conversely, assume that (1) and (2) hold. Let \( A \) and \( B \) be disjoint closed sets. Since \( X \) is strongly zero-dimensional, there exists a clopen set \( U \) such that \( A \subseteq U \) and \( B \subseteq \bar{V} \). Thus, there exists an \( e \in C(X) \) such that \( e(x) = 1 \) for any \( x \in U \) and \( e(x) = 0 \) for any \( x \in X - U \). Clearly, \( e = e^2 \in C(X) \). By hypothesis, we have a projection \( f \in q(X) \) and a unit \( u \in q(X) \) such that \( e = f + u \). In view of Lemma 8, write \( f = = \frac{e}{2} \) with self-adjoint \( a, b \in R \). Since \( e, f \in q(X) \) are idempotents, we see that \( (e-f)^3 = e-f \), and so \( u^2 = 1 \). That is, \( (e-f)^2 = 1 \). This implies that \( e(1-2f) = 1-f \), and so \( e = (1-2f)(1-f) \). This means that \[ e = \frac{(b-2a)(b-a)}{2b^2}, \] and so \( eb^2 = (b-2a)(b-a) \). Since \( a, b \in R \) are self-adjoint, we see that \( e^*b^2 = ((b-2a)(b-a))^* = (b-2a)(b-a) = eb^2 \). But \( b \in N_D(R) \), and so \( e = e^* = e^2 \). Thus, \( U \) is a *-clopen; hence that \( X \) is strongly *-zero-dimensional. According to Lemma 3, we complete the proof.

**Theorem 8.** Let \( R \) be a *-ring, and let \( X \) be a spectrum space of \( R \). Then \( R \) is a strongly *-clean ring if and only if

1. \( R \) is an abelian exchange ring;
2. \( q(X) \) is *-clean.

**Proof.** Since every locally compact Hausdorff space is completely regular, we see that the spectrum space \( X \) of \( R \) is always completely regular.

If \( R \) is a strongly *-clean ring, then \( R \) is an abelian exchange ring. By virtue of Theorem 5, \( C(X) \) is *-clean. In light of Lemma 11, \( q(X) \) is *-clean.

Conversely, assume that (1) and (2) hold. Then \( X \) is strongly zero-dimensional. According to Lemma 11, \( C(X) \) is *-clean. Therefore \( R \) is strongly *-clean by Theorem 5.

**References**


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