THE VORONOVSKAJA TYPE ASYMPTOTIC FORMULA FOR
q-DERIVATIVE OF INTEGRAL GENERALIZATION OF
q-BERNSTEIN OPERATORS

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Abstract. The Voronovskaja type asymptotic formula for function having $q$-
derivative of the integral generalization Bernstein operators based on $q$-integer
is discussed. The same formula for Stancu type generalization of this operators
is mentioned.

1. Introduction

The classical Bernstein-Durrmeyer operators $D_n$ introduced by Durrmeyer [1] is
associated with an integrable function $f$ on the interval $[0, 1]$ and is defined as

$$D_n(f; x) = (n + 1) \sum_{k=0}^{n} p_{n,k}(x) \int_{0}^{1} p_{n,k}(t)f(t)dt, \quad x \in [0, 1], \quad (1.1)$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

These operators have been studied by Derriennic [2] and many others. For the
last 30 years, q-calculus has been an active area of research in approximation theory.
In 1987, the $q$-analogues of Bernstein operators was introduced by Lupas [3] and in
[4], $q$-generalization of the operators (1.1) was introduced as

$$D_{n,q}(f; x) = [n+1]_q \sum_{k=0}^{n} q^{-k} p_{n,k}(q;x) \int_{0}^{1} f(t)p_{n,k}(qt)dt, \quad (1.2)$$

where $p_{n,k}(q;x) = \binom{n}{k}_q x^k (1-x)^{n-k}_q$.

The rate of convergence of the operators (1.2) was discussed by Zeng et al.
[5]. In 2014, Mishra and Patel [6, 7] introduced the generalization due to Stancu
and proved Voronovskaja type asymptotic formula and various other approximation properties of the $q$-Durrmeyer-Stancu operators. Here, in this manuscript, we establish Voronovskaja type asymptotic formula for function having $q$-derivative.

2. Estimation of moments and Asymptotic Formula

In the sequel, we shall need the following auxiliary results:

**Theorem 1** ([8]). If $m$-th ($m > 0, m \in \mathbb{N}$) order moments of operator (1.2) is defined as

$$D_n^q; m(x) = D_n.q(t^m, x) = [n + 1]_q \sum_{k=0}^{n} q^{-k} p_n,k(q; x) \int_{0}^{1} p_n,k(q; t)t^m dt, \quad x \in [0, 1],$$

then $D_n^q; 0(x) = 1$ and for $n > m + 2$, we have the following recurrence relation,

$$[n+m+2]_q D_{n,m+1}^q (x) = ([m+1]_q + q^{m+1} x[n]_q D_{n,m}^q(x)) + x(1-x)q^{m+1}D_q(D_{n,m}^q(x)).$$

To establish asymptotic formula for functions having $q$-derivative, it is necessary to compute moments of first to fourth degree. Using above theorem one can have first, second, third and fourth order moments. The first three moments of Lemma 1 was also established in [4].

**Lemma 1** ([4, 8]). For all $x \in [0, 1], n = 1, 2, \ldots$ and $0 < q < 1$, we have

- $D_n.q(1, x) = 1$;
- $D_n.q(t, x) = \frac{1+q^n n!}{(n+2)_q};$
- $D_n.q(t^2, x) = q^2 [2]_q [n]_q [n+1]_q t^2 (1+q) [n+3]_q [n+2]_q [n+1]_q x$;
- $D_n.q(t^3, x) = q^3 [3]_q [n]_q [n+1]_q [n+2]_q t^3 + q^2 [2]_q [n]_q [n+1]_q (1+q) [n+3]_q [n+2]_q [n+1]_q x$;
- $D_n.q(t^4, x) = \frac{q^4 [4]_q [n]_q [n+1]_q [n+2]_q [n+3]_q t^4}{(n+4)_q (n+3)_q (n+2)_q (n+1)_q} + q^3 [3]_q [n]_q [n+1]_q (1+q) [n+3]_q [n+2]_q [n+1]_q x$;
- $D_n.q(t^5, x) = \frac{q^5 [5]_q [n]_q [n+1]_q [n+2]_q [n+3]_q [n+4]_q t^5}{(n+5)_q (n+4)_q (n+3)_q (n+2)_q (n+1)_q} + q^4 [4]_q [n]_q [n+1]_q (1+q) [n+3]_q [n+2]_q [n+1]_q x$;
- $D_n.q(t^6, x) = \frac{q^6 [6]_q [n]_q [n+1]_q [n+2]_q [n+3]_q [n+4]_q [n+5]_q t^6}{(n+6)_q (n+5)_q (n+4)_q (n+3)_q (n+2)_q (n+1)_q} + q^5 [5]_q [n]_q [n+1]_q (1+q) [n+3]_q [n+2]_q [n+1]_q x$.

**Lemma 2.** For all $x \in [0, 1], n = 1, 2, \ldots$ and $0 < q < 1$, we have

- $D_n.q(t, x) = 1 - (1+q^{n+1}) x$;
- $D_n.q(1-t, x) = q^2 [2]_q [(n+1)q^{-1} (1+q^{n+1}) x^2 + n]_q (1+q^2) [n+3]_q [n+2]_q [n+1]_q x$;
- $D_n.q(t-1, x) = q^3 [3]_q [(n+1)q^{-1} (1+q^{n+1}) x^3 + n]_q (1+q^3) [n+3]_q [n+2]_q [n+1]_q x$;
- $D_n.q(1-t-1, x) = q^4 [4]_q [(n+1)q^{-1} (1+q^{n+1}) x^4 + n]_q (1+q^4) [n+3]_q [n+2]_q [n+1]_q x$;
- $D_n.q(t-1-t-1, x) = q^5 [5]_q [(n+1)q^{-1} (1+q^{n+1}) x^5 + n]_q (1+q^5) [n+3]_q [n+2]_q [n+1]_q x$;
- $D_n.q(1-t-1-t-1, x) = q^6 [6]_q [(n+1)q^{-1} (1+q^{n+1}) x^6 + n]_q (1+q^6) [n+3]_q [n+2]_q [n+1]_q x$.

**Proof.**
Theorem 2. Let $f$ be bounded and integrable on the interval $[0,1]$ and $(q_n)$ denote a sequence such that $0 < q_n < 1$, $q_n \to 1$ and $q_n^2 \to c$ as $n \to \infty$, where $c$ is arbitrary constant. Then we have for a point $x \in (0,1)$,\[
abla n \to \infty [n]_{q_n} |D_{n,q_n}(f;x) - f(x)| = (1 - 2x) \lim_{n \to \infty} D_{q_n} f(x) + x(1 - x) \lim_{n \to \infty} D_{q_n}^2 f(x).
\]

Proof: By $q$-Taylor formula [9] for $f$, we have\[
abla f(t) = f(x) + D_{q_n} f(x)(t-x) + \frac{1}{[2]_{q_n}} D_{q_n}^2 f(x)(t-x)^2_{q_n} + \theta_{q_n}(x; t)(t-x)^2_{q_n},
\]
for $0 < q < 1$, where\[
\theta_{q_n}(x; t) = \begin{cases} 
\frac{f(t) - f(x) - D_{q_n} f(x)(t-x) - \frac{1}{[2]_{q_n}} D_{q_n}^2 f(x)(t-x)^2_{q_n}}{(t-x)^2_{q_n}} & \text{if } x \neq t \\
0, & \text{if } x = t.
\end{cases}
\]

We know that for $n$ large enough\[
\lim_{t \to x} \theta_{q_n}(x; t) = 0.
\]
(2.2)

That is for any $\epsilon > 0$, there exists a $\delta > 0$ such that\[
|\theta_{q_n}(x; t)| \leq \epsilon;
\]
(2.3)

for $|t - x| < \delta$ and $n$ sufficiently large. Using (2.1), we can write\[
D_{n,q_n}(f;x) - f(x) = D_{q_n} f(x) D_{n,q_n}((t-x)_{q_n};x) + \frac{D_{q_n}^2 f(x)}{[2]_{q_n}} D_{n,q_n}((t-x)^2_{q_n};x) + E_{q_n}^n(x),
\]
where\[
E_{q_n}^n(x) = [n+1]_{q_n} \sum_{k=0}^{n} q^{-k} p_{n,k}(q_n; x) \int_0^1 \theta_{q_n}(x; t)p_{n,k}(q_n; q_n t)(t-x)^2_{q_n} d_{q_n} t.
\]

By Lemma 2, we have\[
\lim_{n \to \infty} [n]_{q_n} D_{n,q_n}((t-x)_{q_n};x) = (1 - 2x) \quad \text{and} \quad \lim_{n \to \infty} [n]_{q_n} D_{n,q_n}((t-x)^2_{q_n};x) = 2x(1-x).
\]

In order to complete the proof of the theorem, it is sufficient to show that\[
\lim_{n \to \infty} [n]_{q_n} E_{q_n}^n(x) = 0.
\]
We proceed as follows: Let\[
P_{n,1}(x) = [n]_{q_n} [n+1]_{q_n} \sum_{k=0}^{n} q^{-k} p_{n,k}(q_n; x) \int_0^1 \theta_{q_n}(x; t)p_{n,k}(q_n; q_n t)(t-x)^2_{q_n} \chi_x(t) d_{q_n} t
\]
and\[
P_{n,2}(x) = \quad [n]_{q_n} [n+1]_{q_n} \sum_{k=0}^{n} q^{-k} p_{n,k}(q_n; x) \int_0^1 \theta_{q_n}(x; t)p_{n,k}(q_n; q_n t)(t-x)^2_{q_n}(1 - \chi_x(t)) d_{q_n} t,
\]
so that
\[ [n]_{q_n} E_n^q(x) \leq P_{n,1}^q(x) + P_{n,2}^q(x), \]
where \( \chi_c(t) \) is the characteristic function of the interval \( \{ t : |t - x| < \delta \} \).

It follows from (2.3) that
\[ P_{n,1}^q(x) = 2\varepsilon x (1 - x) \text{ as } n \to \infty. \]

If \( |t - x| \geq \delta \), then \( |\theta_{q_n}(x; t)| \leq M \frac{\varepsilon}{\delta^2} (t - x)^2 \), where \( M > 0 \) is a constant. Since
\[
(t - x)^2 = (t - q_n^2 x + q_n^2 x - x) (t - q_n^2 x + q_n^2 x - x) \\
= (t - q_n^2 x) (t - q_n^3 x) + x(q_n^3 - 1) (t - q_n^2 x) + x(q_n^2 - 1) (t - q_n^2 x) \\
+ x^2 (q_n^2 - q_n^3) + x^2 (q_n^3 - 1) (q_n^2 - 1),
\]
we have
\[
|P_{n,2}^q(x)| \leq \frac{M}{\delta^2} \{ [n]_{q_n} D_{n,q_n} ((t - x)_q^4; x) + x(2 - q_n^2 - q_n^3) [n]_{q_n} D_{n,q_n} ((t - x)_q^3; x) \\
+ x^2 (q_n^2 - 1)^2 [n]_{q_n} D_{n,q_n} ((t - x)_q^2; x) \}.
\]

Using Lemma 2, we have
\[
D_{n,q_n} ((t - x)_q^4; x) \leq \frac{C_1}{[n]_{q_n}^3}, \quad D_{n,q_n} ((t - x)_q^3; x) \leq \frac{C_2}{[n]_{q_n}^2} \text{ and } D_{n,q_n} ((t - x)_q^2; x) \leq \frac{C_3}{[n]_{q_n}},
\]
and the desired result is obtained.

**Corollary 1.** Let \( f \) be bounded and integrable on the interval \([0, 1]\) and \((q_n)\) denote a sequence such that \( 0 < q_n < 1, q_n \to 1 \) and \( q_n^4 \to c \) as \( n \to \infty \), where \( c \) is an arbitrary constant. Suppose that the first and second derivatives \( f'(x) \) and \( f''(x) \) exist at a point \( x \in (0, 1) \). Then, we have, for a point \( x \in (0, 1) \)
\[
\lim_{n \to \infty} [n]_{q_n} [D_{n,q_n} (f; x) - f(x)] = (1 - 2x)f'(x) + x(1 - x)f''(x).
\]

### 3. Asymptotic formula for the Durrmeyer-Stancu operators

In the year 1968, Stancu [10] generalized Bernstein operators and discussed its approximation properties. After that many researchers gave Stancu type generalization of several operators on finite and infinite intervals. We refer the readers to [11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and the references therein. As mentioned in the introduction, Stancu generalization of \( q \)-Durrmeyer operators (1.2) was discussed by Mishra and Patel [6], which is defined as follows:
\[
D_{n,q}^{\alpha, \beta} = [n + 1]_q \sum_{k=0}^{n} q^{-k} p_{n,k}(q; x) \int_0^1 f \left( \frac{[n]_q t + \alpha}{[n]_q + \beta} \right) p_{n,k}(q; qt) d_q t, \quad (3.1)
\]
where \( 0 \leq \alpha \leq \beta \) and \( p_{n,k}(q; x) \) as same as defined in (1.2). We shall need the following lemmas for proving our results.
Lemma 3 ([7]). We have \( D_{n,q}^{\alpha,\beta}(1; x) = 1 \), \( D_{n,q}^{\alpha,\beta}(t; x) = \frac{[n]_q + \alpha[n+2]_q + q[n]_q^2}{[n+2]_q([n]_q + \beta)} \),

\[
D_{n,q}^{\alpha,\beta}(t^2; x) = \left( \frac{q^n[n]_q^2}{[n+2]_q([n]_q + \beta)} - 1 \right) x + \frac{[n]_q + \alpha[n+2]_q}{[n+2]_q([n]_q + \beta)},
\]

\[
D_{n,q}^{\alpha,\beta}((t - x)^3; x) = \frac{q^n[n]_q^4 - 2q[n]_q^3 - 2q^n[n]_q^2([n]_q + \beta) + [n]_q + 2q^n[n+2]_q([n]_q + \beta)}{[n+2]_q([n+3]_q + \beta)^2} x^2
\]

\[
+ \frac{q^n[n]_q^2 - 2q[n]_q^2 + 2q^n[n]_q([n+3]_q - 2q^n[n+2]_q [n]_q + [n]_q + 2q^n[n+2]_q [n]_q + 2q^n[n+2]_q [n]_q + 2q^n[n+2]_q [n]_q) x}{[n+2]_q([n+3]_q + \beta)^2}
\]

\[
+ \frac{(1+q)[n]_q^2 + 2q [n]_q [n+3]_q}{[n+2]_q([n+3]_q + \beta)^2}.
\]

Lemma 4 ([7]). We have

\[
D_{n,q}^{\alpha,\beta}(t - x, x) = \left( \frac{q^n[n]_q^2}{[n+2]_q([n]_q + \beta)} - 1 \right) + \frac{[n]_q + \alpha[n+2]_q}{[n+2]_q([n]_q + \beta)},
\]

\[
D_{n,q}^{\alpha,\beta}((t - x)^3, x) = \frac{q^n[n]_q^4 - 2q[n]_q^3 - 2q^n[n]_q^2([n]_q + \beta) + [n]_q + 2q^n[n+2]_q([n]_q + \beta)}{[n+2]_q([n+3]_q + \beta)^2} x^2
\]

\[
+ \frac{q^n[n]_q^2 - 2q[n]_q^2 + 2q^n[n]_q([n+3]_q - 2q^n[n+2]_q [n]_q + [n]_q + 2q^n[n+2]_q [n]_q + 2q^n[n+2]_q [n]_q) x}{[n+2]_q([n+3]_q + \beta)^2}
\]

\[
+ \frac{(1+q)[n]_q^2 + 2q [n]_q [n+3]_q}{[n+2]_q([n+3]_q + \beta)^2}.
\]

Remark 1 ([7]). For all \( m \in \mathbb{N} \cup \{0\} \), \( 0 \leq \alpha \leq \beta \), we have the following recursive relation for the images of the monomials \( t^m \) under \( D_{n,q}^{\alpha,\beta} \) in terms of \( D_{n,q} \), \( j = 0, 1, 2, \ldots, m \), as

\[
D_{n,q}^{\alpha,\beta}(t^m; x) = \sum_{j=0}^{m} \binom{m}{j} \frac{[n]_q^j \alpha^{m-j}}{([n]_q + \beta)^m} D_{n,q}(t^j, x).
\]

Now, let us compute the moments and central moments of order 3 and 4 for the operators (3.1) in the following manner:

\[
D_{n,q}^{\alpha,\beta}(t^3; x) = \frac{q^n[n]_q^4 + n - 1][n - 2]_q}{([n]_q + \beta)^3} x^3 + \frac{q^n[n]_q^3 + n - 1][n - 1]_q}{([n]_q + \beta)^2} x^2
\]

\[
+ \frac{q^n[n]_q^2 + n - 1][n - 2]_q}{([n]_q + \beta)} x + \frac{q^n[n]_q + n + 4][n + 4]_q}{([n]_q + \beta)} x^n,
\]

Also,

\[
D_{n,q}^{\alpha,\beta}(t^4; x) = \frac{q^n[n]_q^5 + n - 1][n - 2]_q}{([n]_q + \beta)^4} x^4 + \frac{q^n[n]_q^4 + n - 1][n - 2]_q}{([n]_q + \beta)^3} x^3
\]

\[
+ \frac{q^n[n]_q^3 + n - 1][n - 2]_q}{([n]_q + \beta)^2} x^2 + \frac{q^n[n]_q^2 + n - 1][n - 2]_q}{([n]_q + \beta)} x + \frac{q^n[n]_q + n + 4][n + 4]_q}{([n]_q + \beta)} x^n,
\]
Now, using the identity \((t - x)^3 = t^3 - 3t^2x + 3tx^2 - x^3\) and linear properties of the operators \(D_n^{a,b}\), we get

\[
D_n^{a,b}((t-x)^3) = q^3 \left[ \frac{q^2[n]_{q}^3(n-1)_{q} - n(q-1)_{q}}{(n+1)_{q} + n + 2q_{n+3}q}\right] \left[ \frac{q^2[3]_{q}^2(n-1)_{q}}{(n+1)_{q} + n + 2q_{n+3}q}\right] + \frac{q[2]_{q}^2(n-1)_{q}}{(n+1)_{q} + n + 2q_{n+3}q} \right] x^3
\]

Finally, using identity \((t-x)^4 = t^4 - 4t^3x + 6t^2x^2 - 4tx^3 + x^4\), we have

\[
D_n^{a,b}((t-x)^4) = q^4 \left[ \frac{q^2[n]_{q}^4(n-1)_{q} - n(q-1)_{q}}{(n+1)_{q} + n + 2q_{n+3}q}\right] \left[ \frac{q^2[3]_{q}^2(n-1)_{q}}{(n+1)_{q} + n + 2q_{n+3}q}\right] + \frac{q[2]_{q}^2(n-1)_{q}}{(n+1)_{q} + n + 2q_{n+3}q} \right] x^4
\]

Theorem 3. Let \(f\) be bounded and integrable on the interval \([0,1]\) and let \(\{a_n\}\) denote a sequence such that \(0 < a_n < 1\), \(a_n \rightarrow 1\) and \(q_{a_n}^n \rightarrow c\) as \(n \rightarrow \infty\), where \(c\) is arbitrary constant. Then, we have, for a point \(x \in (0,1)\)

\[
\lim_{n \to \infty} [n]_{q_n} D_n^{a,b}(f;x) - f(x) = (1 + \alpha(2-\beta)x) \lim_{n \to \infty} D_n f(x) + x(1-x) \lim_{n \to \infty} D_n^2 f(x).
\]

The proof of the above lemma follows along the lines of the proof of Theorem 2, using Lemma 4 and Remark 1; thus, we omit the details.
Corollary 2 ([6]). Let $f$ be bounded and integrable on the interval $[0,1]$ and let $(q_n)$ denote a sequence such that $0 < q_n < 1$, $q_n \rightarrow 1$ and $q_n \rightarrow c$ as $n \rightarrow \infty$, where $c$ is arbitrary constant. Suppose that the first and second derivatives $f'(x)$ and $f''(x)$ exist at a point $x \in (0,1)$. Then, we have, for a point $x \in (0,1)$,

$$
\lim_{n \rightarrow \infty} [n]_{q_n} [D_n^{\alpha,\beta}(f;x) - f(x)] = (1 + \alpha - (2 + \beta)x)f'(x) + x(1-x)f''(x).
$$

Remark 2. Theorem 2 and Theorem 3, give asymptotic formula for $q$-Durrmeyer operators and $q$-Durrmeyer-Stancu operators respectively. If $f$ has first and second derivatives, then $\lim_{n \rightarrow \infty} D_n f(x) = f'(x)$ and $\lim_{n \rightarrow \infty} D_n^2 f(x) = f''(x)$. We obtain the results of Mishra and Patel [6, Theorem 5], which are mentioned in Corollary 2. So our results are more general than the existing ones.

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