Abstract. The sequence space $BV_\sigma$ was introduced and studied by Mursaleen [9]. In this paper we extend $BV_\sigma$ to $BV_\sigma(M,p,r)$ and study some properties and inclusion relations on this space.

1. Introduction

Let $l_\infty$ and $c$ denote the Banach spaces of bounded and convergent sequences $x = (x_k)_{k=1}^\infty$ respectively. Let $\sigma$ be an injection of the set of positive integers $\mathbb{N}$ into itself having no finite orbits and $T$ be the operator defined on $l_\infty$ by $T((x_n)_{n=1}^\infty) = (x_{\sigma(n)})_{n=1}^\infty$.

A positive linear functional $\phi$, with $\|\phi\| = 1$, is called a $\sigma$ - mean or an invariant mean if $\phi(x) = \phi(Tx)$ for all $x \in l_\infty$.

A sequence $x$ is said to be $\sigma$ - convergent , denoted by $x \in V_\sigma$, if $\phi(x)$ takes the same value, called $\sigma - \lim x$, for all $\sigma$- means $\phi$. We have (see Schaefer [14])

$$V_\sigma = \left\{ x = (x_n) : \sum_{m=1}^{\infty} t_{m,n}(x) = L \text{ uniformly in } n, \ L = \sigma - \lim x \right\},$$

where for $m \geq 0$, $n > 0$

$$t_{m,n}(x) = \frac{x_n + x_{\sigma(n)} + \cdots + x_{\sigma^m(n)}}{m + 1}, \text{ and } t_{-1,n} = 0,$$

where $\sigma^m(n)$ denotes the $m$th iterate of $\sigma$ at $n$. In particular, if $\sigma$ is the translation, a $\sigma$ - mean is often called a Banach limit and $V_\sigma$ reduces to f , the set of almost

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convergent sequences (see Lorentz [5]). Subsequently invariant means have been studied by Ahmad and Mursaleen [1], Mursaleen [8], Raimi [12] and many others.

The concept of paranorm is closely related to linear metric spaces. It is a generalization of that of absolute value. Let $X$ be a linear space. A function $g : X \to \mathbb{R}$ is called paranorm, if

- $g(x) \geq 0$, for all $x \in X$,
- $g(-x) = g(x)$, for all $x \in X$,
- $g(x + y) \leq g(x) + g(y)$, for all $x, y \in X$,
- If $(\lambda_n)$ is a sequence of scalars with $\lambda_n \to \lambda$ ($n \to \infty$) and $(x_n)$ is a sequence of vectors with $g(x_n - x) \to 0$ ($n \to \infty$), then $g(\lambda_n x_n - \lambda x) \to 0$ ($n \to \infty$).

A paranorm $g$ for which $g(x) = 0$ implies $x = 0$ is called a total paranorm on $X$, and the pair $(X, g)$ is called a totally paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (cf. [15, Theorem 10.4.2, p. 183]).

A map $M : \mathbb{R} \to [0, +\infty]$ is said to be an Orlicz function if $M$ is even, convex, left continuous on $\mathbb{R}_+$, continuous at zero, $M(0) = 0$ and $M(u) \to \infty$ as $u \to \infty$. If $M$ takes value zero only at zero we will write $M > 0$ and if $M$ takes only finite values we will write $M < \infty$. [2,3,6,7,10,13].

W. Orlicz [11] used the idea of orlicz function to construct the space $(L^M)$. Lindenstrauss and Tzafriri [4] used the idea of Orlicz function to define orlicz sequence space

$$\ell_M := \left\{ x \in \omega : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty \text{ for some } \rho > 0 \right\}$$

in more detail. $\ell_M$ is a Banach space with the norm

$$||x|| := \inf\{\rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1\}$$

The space $l_M$ is closely related to the space $l_p$, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

The $\triangle_2$ - condition is equivalent to

$$M(Lx) \leq KLM(x), \text{ for all values of } x \geq 0, \text{ and for } L > 1.$$ 

An Orlicz function $M$ can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt,$$

where $\eta$ is known as the kernel of $M$, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, $\eta$ is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$. Note that an Orlicz function
satisfies the inequality

\[ M(\lambda x) \leq \lambda M(x) \text{ for all } \lambda \text{ with } 0 < \lambda < 1. \]

Let \( E \) be a sequence space. Then \( E \) is called

(i) A sequence space \( E \) is said to be symmetric if \( (x_n) \in E \) implies \( (x_{\pi(n)}) \in E \), where \( \pi(n) \) is a permutation of the elements of the elements of \( \mathcal{N} \).

(ii) Solid (or normal), if \( (\alpha_k x_k) \in E \), whenever \( (x_k) \in E \) for all sequences of scalars \( (\alpha_k) \) with \( |\alpha_k| \leq 1 \) for all \( k \in \mathcal{N} \).

\[ \text{Lemma 1.1.} \quad \text{A sequence space } E \text{ is solid implies } E \text{ is monotone.} \]

Mursaleen [9] defined the sequence space

\[ BV_\sigma = \left\{ x \in l_\infty : \sum_{n} |\phi_{m,n}(x)| < \infty, \text{ uniformly in } n \right\}, \]

where

\[ \phi_{m,n}(x) = t_{m,n}(x) - t_{m-1,n}(x) \]

assuming that

\[ t_{m,n}(x) = 0, \text{ for } m = -1. \]

A straightforward calculation shows that

\[ \phi_{m,n}(x) = \begin{cases} \frac{1}{m(m+1)} \sum_{j=1}^{m} j(x_{\sigma^j(n)} - x_{\sigma^{j-1}(n)}) & (m \geq 1) \\ x_n, & (m = 0) \end{cases} \]

Note that for any sequence \( x, y \) and scalar \( \lambda \) we have

\[ \phi_{m,n}(x + y) = \phi_{m,n}(x) + \phi_{m,n}(y) \text{ and } \phi_{m,n}(\lambda x) = \lambda \phi_{m,n}(x). \]
2. Main Results.

Let $M$ be an Orlicz function, $p = (p_m)$ be any sequence of strictly positive real numbers and $r \geq 0$. Now we define the sequence space as follows:

$$BV_\sigma(M, p, r) = \left\{ x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} \left[ M \left( \frac{\phi_{m,n}(x)}{p} \right) \right]^{p_m} < \infty, \right. \left. \text{uniformly in } n \text{ and for some } \rho > 0 \right\}.$$ 

For $M(x) = x$ we get

$$BV_\sigma(p, r) = \left\{ x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} |\phi_{m,n}(x)|^{p_m} < \infty, \text{ uniformly in } n \right\}.$$

For $p_m = 1$, for all $m$, we get

$$BV_\sigma(M, r) = \left\{ x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} \left[ M \left( \frac{\phi_{m,n}(x)}{p} \right) \right] < \infty, \right. \left. \text{uniformly in } n \text{ and for some } \rho > 0 \right\}.$$ 

For $r = 0$ we get

$$BV_\sigma(M, p) = \left\{ x = (x_k) : \sum_{m=1}^{\infty} \left[ M \left( \frac{\phi_{m,n}(x)}{p} \right) \right]^{p_m} < \infty, \right. \left. \text{uniformly in } n \text{ and for some } \rho > 0 \right\}.$$ 

For $M(x) = x$ and $r = 0$ we get

$$BV_\sigma(p) = \left\{ x = (x_k) : \sum_{m=1}^{\infty} |\phi_{m,n}(x)|^{p_m} < \infty, \text{ uniformly in } n \right\}.$$ 

For $p_m = 1$, for all $m$ and $r = 0$ we get

$$BV_\sigma(M) = \left\{ x = (x_k) : \sum_{m=1}^{\infty} \left[ M \left( \frac{\phi_{m,n}(x)}{p} \right) \right] < \infty, \right. \left. \text{uniformly in } n \text{ and for some } \rho > 0 \right\}.$$ 

For $M(x) = x$, $p_m = 1$, for all $m$, and $r = 0$ we get

$$BV_\sigma = \left\{ x = (x_k) : \sum_{m=1}^{\infty} |\phi_{m,n}(x)| < \infty, \text{ uniformly in } n \right\}.$$ 

**Theorem 2.1.** The sequence space $BV_\sigma(M, p, r)$ is a linear space over the field $\mathbb{C}$ of complex numbers.
Proof. Let \( x, y \in BV_\sigma(M, p, r) \) and \( \alpha, \beta \in \mathcal{C} \). Then there exist positive numbers \( \rho_1 \) and \( \rho_2 \) such that
\[
\sum_{m=1}^{\infty} \frac{1}{m^r} \left[ M \left( \frac{\phi_{m,n}(x)}{\rho_1} \right) \right]^{p_m} < \infty
\]
and
\[
\sum_{m=1}^{\infty} \frac{1}{m^r} \left[ M \left( \frac{\phi_{m,n}(y)}{\rho_2} \right) \right]^{p_m} < \infty, \text{ uniformly in } n.
\]
Define \( \rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2) \). Since \( M \) is nondecreasing and convex we have
\[
\sum_{m=1}^{\infty} \frac{1}{m^r} \left[ M \left( \frac{|\alpha \phi_{m,n}(x) + \beta \phi_{m,n}(y)|}{\rho_3} \right) \right]^{p_m}
\leq \sum_{m=1}^{\infty} \frac{1}{m^r} \left[ M \left( \frac{|\alpha \phi_{m,n}(x)|}{\rho_3} \right) + M \left( \frac{|\beta \phi_{m,n}(y)|}{\rho_3} \right) \right]^{p_m}
\leq \sum_{m=1}^{\infty} \frac{1}{m^r} \frac{1}{2} \left[ M \left( \frac{\phi_{m,n}(x)}{\rho_1} \right) + M \left( \frac{\phi_{m,n}(y)}{\rho_2} \right) \right] < \infty, \text{ uniformly in } n.
\]
This proves that \( BV_\sigma(M, p, r) \) is a linear space over the field \( \mathcal{C} \) of complex numbers. \( \square \)

**Theorem 2.2.** For any Orlicz function \( M \) and a bounded sequence \( p = (p_m) \) of strictly positive real numbers, \( BV_\sigma(M, p, r) \) is a paranormed (need not be total paranormed) space with
\[
g(x) = \inf_{n \geq 1} \left\{ \rho \in \mathbb{R}^+ : \left( \sum_{m=1}^{\infty} \frac{1}{m^r} \left[ M \left( \frac{|\phi_{m,n}(x)|}{\rho} \right) \right]^{p_m} \right)^{\frac{1}{p}} \leq 1, \text{ uniformly in } n \right\}.
\]
where \( K = \max(1, \sup p_m) \).

**Proof.** It is clear that \( g(x) = g(-x) \). Since \( M(0) = 0 \), we get
\[
\inf \left\{ \rho \right\} = 0, \text{ for } x = 0.
\]
By using Theorem 1, for \( \alpha = \beta = 1 \), we get
\[
g(x + y) \leq g(x) + g(y).
\]
For the continuity of scalar multiplication let \( l \neq 0 \) be any complex number. Then by the definition we have
\[
g(lx) = \inf_{n \geq 1} \left\{ \rho \in \mathbb{R}^+ : \left( \sum_{m=1}^{\infty} \frac{1}{m^r} \left[ M \left( \frac{|\phi_{m,n}(lx)|}{\rho} \right) \right]^{p_m} \right)^{\frac{1}{p}} \leq 1, \text{ uniformly in } n \right\}
\]
\[ g(lx) = \inf_{n \geq 1} \left\{ (|l|s)_{\frac{n}{|l|}} : \left( \sum_{m=1}^{\infty} \frac{1}{m^r} \left[ M \left( \frac{|\phi_{m,n}(lx)|}{s} \right) \right]^{p_m} \right)^{\frac{1}{p_m}} \leq 1, \right\} \]

where \( s = \frac{p}{|l|} \). Since \( |l|^{p_n} \leq \max(1, |l|^H) \), we have

\[
g(lx) \leq \max(1, |l|^H) \inf_{n \geq 1} \left\{ s_{\frac{n}{|l|}} : \left( \sum_{m=1}^{\infty} \frac{1}{m^r} \left[ M \left( \frac{|\phi_{m,n}(x)|}{s} \right) \right]^{p_m} \right)^{\frac{1}{p_m}} \leq 1, \right\}
\]

\[
= \max(1, |l|^H) g(x)
\]

and therefore \( g(lx) \) converges to zero when \( g(x) \) converges to zero in \( BV_\sigma(M, p, r) \).

Now let \( x \) be a fixed element in \( BV_\sigma(M, p, r) \). There exists \( \rho > 0 \) such that

\[ g(x) = \inf_{n \geq 1} \left\{ \rho_{\frac{n}{|l|}} : \left( \sum_{m=1}^{\infty} \frac{1}{m^r} \left[ M \left( \frac{|\phi_{m,n}(x)|}{\rho} \right) \right]^{p_m} \right)^{\frac{1}{p_m}} \leq 1, \text{ uniformly in } n \right\}. \]

Now

\[ g(lx) = \inf_{n \geq 1} \left\{ \rho_{\frac{n}{|l|}} : \left( \sum_{m=1}^{\infty} \frac{1}{m^r} \left[ M \left( \frac{|\phi_{m,n}(lx)|}{\rho} \right) \right]^{p_m} \right)^{\frac{1}{p_m}} \leq 1, \text{ uniformly in } n \right\} \to 0, \]

as \( l \to 0 \).

This completes the proof.

**Theorem 2.3.** Suppose that \( 0 < p_m \leq t_m < \infty \) for each \( m \in \mathbb{K} \) and \( r \geq 0 \). Then

(i) \( BV_\sigma(M, p) \subseteq BV_\sigma(M, t) \),

(ii) \( BV_\sigma(M) \subseteq BV_\sigma(M, r) \).

**Proof.** (i) Suppose that \( x \in BV_\sigma(M, p) \). This implies that

\[ \left[ M \left( \frac{|\phi_{i,n}(x)|}{\rho} \right) \right]^{p_m} \leq 1 \]

for sufficiently large values of \( i \), say \( i \geq m_0 \) for some fixed \( m_0 \in \mathbb{K} \). Since \( M \) is non-decreasing, we have

\[
\sum_{m=m_0}^{\infty} \left[ M \left( \frac{|\phi_{i,n}(x)|}{\rho} \right) \right]^{t_m} \leq \sum_{m=m_0}^{\infty} \left[ M \left( \frac{|\phi_{i,n}(x)|}{\rho} \right) \right]^{p_m} < \infty.
\]

Hence \( x \in BV_\sigma(M, t) \).

The proof of [ii] is trivial.

\[ \square \]
The following result is a consequence of the above result.

**Corollary 1.** If \( 0 < p_m \leq 1 \) for each \( m \), then \( BV_\sigma(M, p) \subseteq BV_\sigma(M) \).

If \( p_m \geq 1 \) for all \( m \), then \( BV_\sigma(M) \subseteq BV_\sigma(M, p) \).

**Theorem 2.4.** The sequence space \( BV_\sigma(M, p, r) \) is solid.

**Proof.** Let \( x \in BV_\sigma(M, p, r) \). This implies that

\[
\sum_{m=1}^{\infty} m^{-r} \left[ M \left( \frac{|\phi_{k,n}(x)|}{p} \right) \right]^{p_m} < \infty.
\]

Let \( (\alpha_m) \) be sequence of scalars such that \( |\alpha_m| \leq 1 \) for all \( m \in \mathbb{N} \). Then the result follows from the following inequality

\[
\sum_{m=1}^{\infty} m^{-r} \left[ M \left( \frac{|\alpha_m \phi_{k,n}(x)|}{p} \right) \right]^{p_m} \leq \sum_{m=1}^{\infty} m^{-r} \left[ M \left( \frac{|\phi_{k,n}(x)|}{p} \right) \right]^{p_m} < \infty.
\]

Hence \( \alpha x \in BV_\sigma(M, p, r) \) for all sequences of scalars \( (\alpha_m) \) with \( |\alpha_m| \leq 1 \) for all \( m \in \mathbb{N} \) whenever \( x \in BV_\sigma(M, p, r) \).

From Theorem 4 and Lemma we have:

**Corollary 2.** The sequence space \( BV_\sigma(M, p, r) \) is monotone.

**Theorem 2.5.** Let \( M_1, M_2 \) be Orlicz functions satisfying \( \Delta_2 \) - condition and \( r, r_1, r_2 \geq 0 \). Then we have

(i) If \( r > 1 \) then \( BV_\sigma(M_1, p, r) \subseteq BV_\sigma(M_0 M_1, p, r) \),

(ii) \( BV_\sigma(M_1, p, r) \cap BV_\sigma(M_2, p, r) \subseteq BV_\sigma(M_1 + M_2, p, r) \),

(iii) If \( r_1 \leq r_2 \) then \( BV_\sigma(M, p, r_1) \subseteq BV_\sigma(M, p, r_2) \).

**Proof.** Since \( M \) is continuous at 0 from right, for \( \epsilon > 0 \) there exists \( 0 < \delta < 1 \) such that \( 0 \leq c \leq \delta \) implies \( M(c) < \epsilon \). If we define

\[
I_i = \left\{ m \in \mathbb{N} : M_1 \left( \frac{|\phi_{m,n}(x)|}{p} \right) \leq \delta \text{ for some } p > 0 \right\} ,
\]

where \( \phi_{m,n}(x) = \frac{\phi_{m,n}(x)}{p} \) and \( \phi_{m,n}(x) = \frac{\phi_{m,n}(x)}{p} \).
\[
I_2 = \left\{ m \in \mathcal{X} : M_1 \left( \frac{|\phi_{m,n}(x)|}{p} \right) > \delta \text{ for some } \rho > 0 \right\},
\]
then, when \( M_1 \left( \frac{|\phi_{m,n}(x)|}{p} \right) > \delta \) we get
\[
M \left( M_1 \left( \frac{|\phi_{m,n}(x)|}{p} \right) \right) \leq \{2M(1)/\delta\} M_1 \left( \frac{|\phi_{m,n}(x)|}{p} \right).
\]
Hence for \( x \in BV_\sigma(M_1, p, r) \) and \( r > 1 \)
\[
\sum_{m=1}^{\infty} m^{-r} \left[ M_0 M_1 \left( \frac{|\phi_{m,n}(x)|}{p} \right) \right]^{p_m} = \sum_{m \in I_1} m^{-r} \left[ M_0 M_1 \left( \frac{|\phi_{m,n}(x)|}{p} \right) \right]^{p_m}
+ \sum_{m \in I_2} m^{-r} \left[ M_0 M_1 \left( \frac{|\phi_{m,n}(x)|}{p} \right) \right]^{p_m}
\leq \sum_{m \in I_1} m^{-r} \left[ e^{p_m} \right]^{p_m}
+ \sum_{m \in I_2} m^{-r} \left[ \{2M(1)/\delta\} M_1 \left( \frac{|\phi_{m,n}(x)|}{p} \right) \right]^{p_m}
\leq \max(e^h, e^H) \sum_{m=1}^{\infty} m^{-r}
+ \max \left( \{2M(1)/\delta\}^h, \{2M(1)/\delta\}^H \right)
\]
(where \( 0 < h = \inf p_m \leq p_m \leq H = \sup p_m < \infty \)).

[ii] The proof follows from the following inequality
\[
m^{-r} \left[ M_1 + M_2 \left( \frac{|\phi_{m,n}(x)|}{p} \right) \right]^{p_m} \leq C m^{-r} \left[ M_1 \left( \frac{|\phi_{m,n}(x)|}{p} \right) \right]^{p_m}
+ C m^{-r} \left[ M_2 \left( \frac{|\phi_{m,n}(x)|}{p} \right) \right]^{p_m}.
\]

[iii] The proof is straightforward.

\[\square\]

**Corollary 3.** Let \( \sigma \) be an Orlicz function satisfying \( \triangle_2 \) - condition. Then we have

1. If \( r > 1 \), then \( BV_\sigma(p, r) \subseteq BV_\sigma(M, p, r) \),
2. \( BV_\sigma(M, p) \subseteq BV_\sigma(M, p, r) \),
3. \( BV_\sigma(p) \subseteq BV_\sigma(p, r) \),
4. \( BV_\sigma(M) \subseteq BV_\sigma(M, r) \).

The proof is straightforward.
ÖZET: $BV_{p}$ dizisi uzayı, Mursaleem tarafından tanımlanmış ve incelenmiştir [9]. Bu çalışmada $BV_{p}$ uzayının, $BV_{p}(M, p, r)$ uzayına genişleterek bu uzaya ilişkin bazı özellikleri ve kapsama bağlamını elde ettik.

References

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